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A BASIS-FREE FORMULA FOR TIME RATE OF HILL'S STRAIN TENSORS

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Abstract—In this paper we use the "π-method" to obtain a basis-free formula for the time rate $\dot{\mathbf{E}}$ of Hill's strain tensor $\mathbf{E}(\mathbf{U}) = \Sigma_i f(\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i$; here $\mathbf{U}$ is the right stretch tensor, λ_i are its eigenvalues and $\{\mathbf{N}_i\}$ is a Lagrangian triad of orthonormal eigenvectors subordinate to $\{\lambda_i\}$; $f(\cdot)$ is a smooth strictly-increasing scalar function that satisfies: $f(1) = 0, f'(1) = 1$. Our formula is generally valid, provided that $\mathbf{U}(\cdot)$ is a C^1 function of time and $f(\cdot)$ is of class C^3. Until now all other basis-free formulae for $\dot{\mathbf{E}}$ in the literature have been burdened by one or other of the following deficiencies: (i) valid only under various special circumstances; (ii) a complete proof is wanting. The formula derived herein is free of these difficulties.

1. INTRODUCTION

Hill (1968, 1981) proposed a class of strain tensors in solid mechanics of finite deformations:

$$\mathbf{E}(\mathbf{U}) = \Sigma_i f(\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i, \tag{1}$$

where

$$\mathbf{U} = \Sigma_i \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i \tag{2}$$

is the right stretch tensor, λ_i are its eigenvalues (the principal stretches), and $\{\mathbf{N}_i\}$ is a Lagrangian triad of orthonormal eigenvectors subordinate to $\{\lambda_i\}$; $f(\cdot)$ is a smooth strictly-increasing scalar function that satisfies: $f(1) = 0, f'(1) = 1$. Hill's class (1) is broad enough to include almost all the commonly used Lagrangian strain tensors. From eqn (1) it is clear that $\mathbf{E}(\cdot)$ is a well-defined isotropic function of $\mathbf{U}$.

Once $\mathbf{E}$ is proposed, there arises naturally the problem of finding an expression for its time rate $\dot{\mathbf{E}}$. Indeed Hill himself derived the principal component formula for $\dot{\mathbf{E}}$ [*cf.* Guo and Dubey (1984) for a more compact form of this formula; see also Scheidler (1991a)]. Evidently Hill's formula, as given in the principal frame $\{\mathbf{N}_i\}$ of $\mathbf{U}$, is valid only in that frame. There remains the problem to find a basis-free formula for $\dot{\mathbf{E}}$. As we shall explain in detail below, while this problem has been tackled by other researchers, until now all the basis-free formulae for $\dot{\mathbf{E}}$ in the literature have suffered from either of the following two difficulties: (i) valid only under various special circumstances; (ii) a complete proof is wanting. Our objective in this paper is to find a basis-free formula for $\dot{\mathbf{E}}$, which is free of the aforementioned deficiencies. In order that $\dot{\mathbf{E}}$ be well defined, we shall henceforth assume that $\mathbf{U}(\cdot)$ is a C^1 function of time t in an interval $\mathscr{T}$, and that the function f in eqn (1) is at least of class C^1 (*cf.* Section 2 below).

Carlson and Hoger (1986a) became the first to obtain a basis-free expression for the derivative $\mathrm{D}\mathbf{E}$ of $\mathbf{E}$, from which a basis-free formula for $\dot{\mathbf{E}}$ of course follows immediately. In their derivation Carlson and Hoger assume that the function f is of class C^7 (*cf.* Remark 7.2 below). Asserting that the nonuniqueness of the orthonormal basis $\{\mathbf{N}_i\}$ when $\mathbf{U}$ has repeated eigenvalues "has been a major source of difficulty" in obtaining the derivative of $\mathbf{E}$, they abandon eqn (1) and use a representation of $\mathbf{E}$ in terms of the eigenprojections of $\mathbf{U}$.

In another paper Hoger (1986), by using the expressions from Carlson and Hoger (1986a), derived explicit formulae for the time rates of change of the logarithmic strain tensors.

In the present paper, we shall derive another basis-free formula for $\dot{\mathbf{E}}$ under the assumption that the function f in eqn (1) is of class C^3. In our derivation we do not use eigenprojections but follow another approach called the "π-method".

The π-method has been used successfully in the derivation of basis-free formulae for other tensor-valued functions of tensors [*cf.* Guo *et al.* (1991a), Guo *et al.* (1992), Guo and Man (1992) and Guo (1993)]. It may be outlined as follows: find the tensor equation or equations the sought tensor should fulfil; confirm the isotropy of the sought tensor function; use the representation theorems for isotropic tensor functions; under the principal frame reduce the problem to solving algebraic equations and use an algorithm based on the fundamental theorem of symmetric polynomials to obtain the final basis-free result by replacing eigenvalues of symmetric tensors, as far as possible, with the corresponding principal invariants. Unlike Carlson and Hoger's method of eigenprojections, the π-method in an intermediary step does follow the work of Hill and take full advantage of the principal frame; the final result, however, will be basis-free. For this reason it is called the "*P*rincipal *A*xis *I*ntrinsic method" (Guo *et al.*, 1991b) or, for brevity, the π-method (Guo, 1992), where the symbol "π" stands for its phonetic equivalent "PAI".

After some mathematical preliminaries, we use the π-method to obtain in Sections 3–5 three different basis-free expressions for $\dot{\mathbf{E}}$, which are valid over the interior $\mathscr{T}_i^0$ of the subsets of time $\mathscr{T}_i$ ($i = 1, 2, 3$) where the principal stretches are distinct, doubly coalescent, and triply coalescent, respectively. In deriving these three expressions, we make no further assumption of smoothness than f to be of class C^1. We derive in Section 6 further basis-free expressions that cover the remaining instants in $\mathscr{T}\backslash(\mathscr{T}_1^0 \cup \mathscr{T}_2^0 \cup \mathscr{T}_3^0)$. There we shall require f to be of class C^3. The expressions derived in Sections 3–6 together give a basis-free formula for $\dot{\mathbf{E}}$. This formula is generally valid, provided that $\mathbf{U}(\cdot)$ is a C^1 function of time and $f(\cdot)$ is of class C^3. We call ours a complete formula to distinguish it from those in the literature which hold only under various special circumstances (*cf.* Remark 2.1). Carlson and Hoger's (1986a) formula is also complete; there is, however, a gap in their proof of the formula (*cf.* Remark 7.2), and they impose the smoothness assumption that f is of class C^7. We believe that the gap in their proof can be closed and that Carlson and Hoger's assumption of smoothness on f can be weakened. Both beliefs, however, remain to be substantiated by further work.

Our formula for the case of distinct stretches has five scalar coefficients α_p. In Section 7 we show that the domains of the functions α_p ($p = 1, \ldots, 5$) may be extended by continuity to allow for coalescence of eigenvalues of $\mathbf{U}$. With the functions α_p thus extended, the formula for the case of distinct eigenvalues in fact encompasses all the expressions that pertain to the other possibilities. Thus we have our formula written in a compact form in eqn (134).

After completing in spring 1990 a draft that contained Sections 3–5 of the present paper, several papers (Scheidler, 1991a,b, 1992; Wang and Duan, 1991) treating different aspects of the same topic have come to our attention. The formula given by Wang and Duan is what we regard as incomplete; further comments in this regard will be given in Remark 2.1. Two of Scheidler's papers study component formulae and approximate basis-free formulae, respectively. In the third paper (Scheidler, 1992), he gave in one section a preview of a paper in preparation, where he would treat exact basis-free formulae. It is clear from the preview that Scheidler will use eigenprojections in his forthcoming paper. We expect his paper will hardly overlap ours either in method or in results.

Our present approach without major change can be used for strain tensors of Eulerian type obtained by replacing $\mathbf{U}$ in eqn (1) with the left stretch tensor $\mathbf{V}$.

2. MATHEMATICAL PRELIMINARIES

Let $\mathbb{R}^+$ be the positive reals, $\mathscr{T}$ be an interval of time, P_{sym} be the set of symmetric positive-definite tensors, and $\mathbf{U}: \mathscr{T} \to P_{\text{sym}}$ be of class C^1. By a theorem due to Rellich (1969) [*cf.* also Kato (1982)], we know that there are three C^1 functions $\lambda_i: \mathscr{T} \to \mathbb{R}^+$

($i = 1, 2, 3$) which represent the repeated eigenvalues of $\mathbf{U}(t)$ for each $t \in \mathscr{T}$. On the other hand, even if $\mathbf{U}$ is of the class C^{∞}, there need not exist such $\lambda_i(\cdot)$ that are of class C^2. A counter-example to that effect can be easily obtained by modifying the one due to Wasow [*cf.* Kato (1982)]. Henceforth we assume a specific triad of C^1 functions $\lambda_i(\cdot)$ has been chosen.

As far as properties of continuity and differentiability are concerned, the eigenvectors of $\mathbf{U}$ can behave even more erratically than the repeated eigenvalues $\lambda_i(\cdot)$. Even if $\mathbf{U}$ is of class C^{∞}, it need not possess a Lagrangian triad of orthonormal eigenvectors $\{\mathbf{N}_i\}$ that are continuous on $\mathscr{T}$. A counter-example can easily be constructed by slightly modifying the one due to Rellich [*cf.* Kato (1982)], as Scheidler (1991a) did. Both the eigenvalues and eigenvectors of $\mathbf{U}(\cdot)$, however, will be analytic if $\mathbf{U}(\cdot)$ is analytic [*cf.* Rellich (1969)].

Remark 2.1. Since $\mathbf{U}(\cdot)$ generally does not have a continuous Lagrangian triad of orthonormal eigenvectors $\{\mathbf{N}_i(\cdot)\}$, to avoid pitfalls one must be careful in working with them. Wang and Duan (1991) recently derived the following "absolute representation" of $\dot{\mathbf{E}}$:

$$\dot{\mathbf{E}} = \tfrac{1}{2}[\mathbf{g}(\mathbf{U})\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{g}(\mathbf{U})] - \tfrac{1}{2}[\mathbf{g}(\mathbf{U})(\mathbf{\Omega}\mathbf{U} - \mathbf{U}\mathbf{\Omega}) + (\mathbf{\Omega}\mathbf{U} - \mathbf{U}\mathbf{\Omega})\mathbf{g}(\mathbf{U})] + \mathbf{\Omega}\mathbf{E} - \mathbf{E}\mathbf{\Omega}, \qquad (3)$$

where $\mathbf{g}(\mathbf{U}) = \Sigma_i f'(\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i$ and $\mathbf{\Omega}$ is the twirl tensor of $\mathbf{U}$ formally defined by

$$\dot{\mathbf{N}}_i = \mathbf{\Omega}\mathbf{N}_i \quad (i = 1, 2, 3). \qquad (4)$$

Since $\mathbf{E}$ and $\mathbf{g}(\mathbf{U})$ can be expressed in terms of the eigenprojections of $\mathbf{U}$ and there are well-defined basis-free expressions for $\mathbf{\Omega}$ over the subsets $\mathscr{T}_i^0$ [*cf.* Guo *et al.* (1992)], Wang and Duan's formula may indeed be taken as basis-free on those subsets of $\mathscr{T}$. However, unless $\mathbf{U}(\cdot)$ is analytic, eqn (4) and hence $\mathbf{\Omega}$ need not make sense outside the open subsets $\mathscr{T}_i^0$ of $\mathscr{T}$. Indeed, examples [*cf.* Scheidler (1991a), Guo *et al.* (1992)] can be constructed in which some entries of $\mathbf{\Omega}$ tend to infinity as t approaches a boundary point of $\mathscr{T}_1^0$ or $\mathscr{T}_2^0$. Hence, unless $\mathbf{U}(\cdot)$ is analytic, Wang and Duan's formula ia generally valid only on the open subsets $\mathscr{T}_i^0$, not for the entire interval of time $\mathscr{T}$. That $\mathbf{U}$ be analytic in t is far too restrictive to be acceptable as a general assumption in solid mechanics. Hence Wang and Duan's formula must be regarded as an incomplete one. □

Let $\mathbb{R}$ denote the reals. Let $\mathscr{T}_1$, $\mathscr{T}_2$ and $\mathscr{T}_3$ be the subsets of $\mathscr{T}$ on which the eigenvalues of $\mathbf{U}$ are distinct, doubly coalescent, and triply coalescent, respectively. We use the subspace topology of $\mathscr{T}$ in $\mathbb{R}$. The interior $\mathscr{T}_i^0$ of $\mathscr{T}_i$ ($i = 1, 2, 3$) is open in $\mathscr{T}$ and, if nonempty, is a disjoint union of (relatively) open intervals. If $\mathbf{U}$ is of class C^n ($1 \leqslant n \leqslant \infty$) on $\mathscr{T}$, then over each such open interval, the eigenvalues $\lambda_i(\cdot)$ are of class C^n and there exists a subordinate Lagrangian triad of orthonormal eigenvectors $\{\mathbf{N}_i(\cdot)\}$ that are also of class C^n. See Guo *et al.* (1992) for a simple proof of the preceding assertion, which is also subsumed under a more general theorem due to Nomizu [*cf.* Kato (1982)].

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a strictly-increasing function that satisfies $f(1) = 0$ and $f'(1) = 1$. By appealing to a theorem due to Ball (1984), Scheidler (1991a) has shown that the strain tensor $\mathbf{E}(\cdot)$, as defined by eqn (1), is of class C^1 if the function f is of class C^1. We shall henceforth assume that f is at least of class C^1. Hence the composite function $t \mapsto \mathbf{U}(t) \mapsto \mathbf{E}(\mathbf{U}(t))$ is of class C^1, and $\dot{\mathbf{E}}$ is well defined on the interval of time $\mathscr{T}$.

3. CASE OF DISTINCT STRETCHES

In this section we use the π-method to derive a basis-free expression of $\dot{\mathbf{E}}$ for the instants $t \in \mathscr{T}_1^0$. By definition, $\mathscr{T}_1^0$ is the interior of the subset $\mathscr{T}_1$ of instants t at which the eigenvalues of $\mathbf{U}$ are distinct. Since $\mathscr{T}_1^0$ is open in $\mathscr{T}$ and all our considerations are local in time, we shall simply treat $\mathscr{T}_1^0$, with no loss in generality, as if it is an interval.

Henceforth we assume that a specific triad of eigenvalues $\{\lambda_i(\cdot)\}$ and a subordinate Lagrangian triad $\{\mathbf{N}_i(\cdot)\}$ have been chosen; for each i, $\lambda_i(\cdot)$ and $\mathbf{N}_i(\cdot)$ are C^1 functions of time t on $\mathscr{T}_1^0$. To describe the rotation of $\mathbf{N}_i$, we may now legitimately write

$$\dot{\mathbf{N}}_i = \mathbf{\Omega}\mathbf{N}_i, \tag{5}$$

where the skew-symmetric

$$\mathbf{\Omega} = \Sigma_{i,j}\, \omega_{ij}\mathbf{N}_i \otimes \mathbf{N}_j \tag{6}$$

is the twirl tensor [*cf.* Guo *et al.* (1992)] for the properties of $\mathbf{\Omega}$. Differentiating eqn (2) with respect to t, we find that $\mathbf{\Omega}$, as a function of $\mathbf{U}$ and $\dot{\mathbf{U}}$, satisfies the tensor equation

$$\mathbf{U\Omega} - \mathbf{\Omega U} = \Sigma_i\, \dot{\lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i - \dot{\mathbf{U}}. \tag{7}$$

If we decompose $\dot{\mathbf{U}}$ in $\{\mathbf{N}_i\}$ and denote its components by $\dot{U}_{ij}$ (here for $\dot{U}_{ij}$ and later on for $\dot{E}_{ij}$, the dot should not be understood as $\mathrm{d}/\mathrm{d}t$):

$$\dot{\mathbf{U}} = \Sigma_{i,j}\, \dot{U}_{ij}\mathbf{N}_i \otimes \mathbf{N}_j, \tag{8}$$

then from eqn (7) we have

$$\dot{U}_{ii} = \dot{\lambda}_i, \quad (i = 1, 2, 3) \tag{9}$$

and

$$\omega_{ij}(\lambda_i - \lambda_j) = -\dot{U}_{ij}, \quad (i \neq j). \tag{10}$$

Since

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1 \tag{11}$$

for the case at hand, we deduce from eqn (10) the components of $\mathbf{\Omega}$ in principal representation:

$$\omega_{ij} = \frac{-\dot{U}_{ij}}{\lambda_i - \lambda_j}, \quad (i \neq j). \tag{12}$$

Now, we differentiate eqn (1) with respect to t:

$$\begin{aligned}\dot{\mathbf{E}}(\mathbf{U};\dot{\mathbf{U}}) &= \mathbf{DE}(\mathbf{U})[\dot{\mathbf{U}}] \\ &= \Sigma_i\, [f_{,i}\dot{\lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i + f_i(\dot{\mathbf{N}}_i \otimes \mathbf{N}_i + \mathbf{N}_i \otimes \dot{\mathbf{N}}_i)] \\ &= \Sigma_i\, f_{,i}\dot{\lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i + \mathbf{\Omega E} - \mathbf{E\Omega},\end{aligned} \tag{13}$$

where $f_i \equiv f(\lambda_i), f_{,i} \equiv f'(\lambda_i)$. Under the principal frame $\{\mathbf{N}_i\}$,

$$\dot{\mathbf{E}} = \Sigma_{i,j}\, \dot{E}_{ij}\mathbf{N}_i \otimes \mathbf{N}_j. \tag{14}$$

Making use of eqns (9), (12) and (14), we can write eqn (13) in its principal componential form:

$$\dot{E}_{ii} = f_{,i}\dot{U}_{ii}, \quad (i = 1, 2, 3), \tag{15}$$

$$\dot{E}_{ij} = -\omega_{ij}(f_i - f_j) = \frac{f_i - f_j}{\lambda_i - \lambda_j}\dot{U}_{ij}, \quad (i \neq j). \tag{16}$$

The strain tensor $\mathbf{E}$, as defined in eqn (1), is an isotropic function of $\mathbf{U}$. Hence, its time derivative $\dot{\mathbf{E}}(\mathbf{U};\dot{\mathbf{U}}) = D\mathbf{E}(\mathbf{U})[\dot{\mathbf{U}}]$ is an isotropic function of $\mathbf{U}$ and $\dot{\mathbf{U}}$, linear in $\dot{\mathbf{U}}$ [*cf.* Guo (1988)]. Rivlin and Ericksen's representation theorem for symmetric-tensor-valued isotropic function of two symmetric tensors [*cf.* Rivlin and Ericksen (1955), Rivlin (1955) and Wang (1970)] suggests that an isotropic function such as $\dot{\mathbf{E}}(\mathbf{U};\dot{\mathbf{U}})$ has the following representation:

$$\dot{\mathbf{E}}(\mathbf{U};\dot{\mathbf{U}}) = \alpha_1\mathbf{I}+\alpha_2\mathbf{U}+\alpha_3\mathbf{U}^2+\alpha_4\dot{\mathbf{U}}+\alpha_5(\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U})+\alpha_6(\mathbf{U}^2\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}^2), \tag{17}$$

where α_4, α_5, α_6 are functions of the principal invariants of $\mathbf{U}$:

$$\left.\begin{aligned} \mathrm{I} &= \lambda_1+\lambda_2+\lambda_3 \\ \mathrm{II} &= \lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2 \\ \mathrm{III} &= \lambda_1\lambda_2\lambda_3 \end{aligned}\right\}, \tag{18}$$

and α_1, α_2, α_3 are functions of I, II, III and three common invariants of $\mathbf{U}$ and $\dot{\mathbf{U}}$:

$$\left.\begin{aligned} J_0 &= \operatorname{tr}\dot{\mathbf{U}} = \dot{\lambda}_1+\dot{\lambda}_2+\dot{\lambda}_3 \\ J_1 &= \operatorname{tr}(\mathbf{U}\dot{\mathbf{U}}) = \lambda_1\dot{\lambda}_1+\lambda_2\dot{\lambda}_2+\lambda_3\dot{\lambda}_3 \\ J_2 &= \operatorname{tr}(\mathbf{U}^2\dot{\mathbf{U}}) = \lambda_1^2\dot{\lambda}_1+\lambda_2^2\dot{\lambda}_2+\lambda_3^2\dot{\lambda}_3 \end{aligned}\right\}. \tag{19}$$

Noll (1955), who referred to Rivlin and Ericksen's representation theorem mentioned above, first wrote down representation formula (17) for symmetric-tensor-valued isotropic functions such as $\dot{\mathbf{E}}(\mathbf{U};\dot{\mathbf{U}})$; he expressed the coefficients α_1, α_2 and α_3 of this formula in the form given in eqn (40) below. Here we need not bother about proving the general validity of representation formula (17), because we shall show constructively that $\dot{\mathbf{E}}$ has indeed such a representation for the case at hand. We rewrite eqn (19) as

$$\begin{pmatrix} J_0 \\ J_1 \\ J_2 \end{pmatrix} = M^{\mathrm{T}}\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix}, \tag{20}$$

where M is the van der Monde matrix

$$M := \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}. \tag{21}$$

Because $\dot{\mathbf{E}}$ is linear in $\dot{\mathbf{U}}$, α_1, α_2 and α_3 are linear in J_0, J_1 and J_2, and can be presented as

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = AM^{\mathrm{T}}\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix}, \tag{22}$$

where the matrix A is

$$A := (\alpha_{ij}). \tag{23}$$

Anticipating that $\dot{\mathbf{E}}$ could be represented as shown in eqn (17), we compare the componential form of eqn (17) with eqns (15) and (16), and obtain

$$\alpha_1+\alpha_2\lambda_i+\alpha_3\lambda_i^2+\alpha_4\dot{\lambda}_i+2\alpha_5\lambda_i\dot{\lambda}_i+2\alpha_6\lambda_i^2\dot{\lambda}_i = f_{,i}\dot{\lambda}_i \quad (i = 1,2,3), \tag{24}$$

$$\alpha_4\dot{U}_{ij}+\alpha_5(\lambda_i+\lambda_j)\dot{U}_{ij}+\alpha_6(\lambda_i^2+\lambda_j^2)\dot{U}_{ij} = \frac{f_i-f_j}{\lambda_i-\lambda_j}\dot{U}_{ij}, \quad (i \neq j), \tag{25}$$

where we have appealed to eqn (9) and put $\dot{\lambda}_i$ for $\dot{U}_{ii}$. This anticipation will be realized if we can show that eqns (24) and (25) indeed have a solution in α_i $(i = 1,\ldots,6)$ which are functions of the appropriate variables given after eqn (17).

Since $\dot{\mathbf{U}}$ can be arbitrary, the linear system (25) is equivalent to

$$\begin{pmatrix} 1 & \lambda_2+\lambda_3 & \lambda_2^2+\lambda_3^2 \\ 1 & \lambda_3+\lambda_1 & \lambda_3^2+\lambda_1^2 \\ 1 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 \end{pmatrix}\begin{pmatrix}\alpha_4\\ \alpha_5\\ \alpha_6\end{pmatrix} = \begin{bmatrix} \dfrac{f_2-f_3}{\lambda_2-\lambda_3} \\ \dfrac{f_3-f_1}{\lambda_3-\lambda_1} \\ \dfrac{f_1-f_2}{\lambda_1-\lambda_2} \end{bmatrix}. \tag{26}$$

By virtue of the inequalities (11), the determinant of the coefficient matrix

$$\Delta = (\lambda_2-\lambda_3)(\lambda_3-\lambda_1)(\lambda_1-\lambda_2) \neq 0, \tag{27}$$

so the system (26) has a unique solution:

$$\left.\begin{aligned} \alpha_4 &= \frac{1}{\Delta}\Sigma_i f_i(\lambda_j^2-\lambda_k^2) \\ \alpha_5 &= \frac{-1}{\Delta}\Sigma_i f_i(\lambda_j-\lambda_k) \\ \alpha_6 &= 0 \end{aligned}\right\}. \tag{28}$$

Here and henceforth, the summation Σ_i is to be carried out for all even permutations (i, j, k) of $(1,2,3)$.

Substituting eqn (28) back into eqn (24), we have

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\end{pmatrix}, \tag{29}$$

where

$$\mu_i \equiv f_{,i}\dot{\lambda}_i-\alpha_4\dot{\lambda}_i-2\alpha_5\lambda_i\dot{\lambda}_i. \tag{30}$$

Solution of eqn (29) reduces to finding the matrix A in eqn (23). In fact, denoting the matrices

$$F = \text{diag.}\,(f_{,1}, f_{,2}, f_{,3}), \quad E = \text{diag.}(1,1,1), \tag{31}$$

$$U = \text{diag.}\,(\lambda_1, \lambda_2, \lambda_3), \tag{32}$$

we can write eqn (29) in the matrix form:

$$M\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}=\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\end{pmatrix}=(F-\alpha_4E-2\alpha_5U)\begin{pmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\end{pmatrix}. \tag{33}$$

Substituting eqn (22) into the left-hand side of eqn (33), by virtue of the arbitrariness of $\{\dot{\lambda}_i\}$, we have

$$MAM^{\mathrm{T}}=F-\alpha_4E-2\alpha_5U,$$

or

$$A=M^{-1}(F-\alpha_4E-2\alpha_5U)M^{-\mathrm{T}}. \tag{34}$$

It is readily seen that A is symmetric and can be expressed as

$$A=A^{(f)}-\alpha_4A^{(0)}-2\alpha_5A^{(1)}, \tag{35}$$

where the matrices

$$\left.\begin{aligned}A^{(f)}&:=M^{-1}FM^{-\mathrm{T}}\\ A^{(0)}&:=M^{-1}M^{-\mathrm{T}}\\ A^{(1)}&:=M^{-1}UM^{-\mathrm{T}}\end{aligned}\right\} \tag{36}$$

are symmetric. Having

$$M^{-1}=\frac{1}{\Delta}\begin{pmatrix}\lambda_2\lambda_3(\lambda_3-\lambda_2) & \lambda_3\lambda_1(\lambda_1-\lambda_3) & \lambda_1\lambda_2(\lambda_2-\lambda_1)\\ \lambda_2^2-\lambda_3^2 & \lambda_3^2-\lambda_1^2 & \lambda_1^2-\lambda_2^2\\ \lambda_3-\lambda_2 & \lambda_1-\lambda_3 & \lambda_2-\lambda_1\end{pmatrix} \tag{37}$$

and using eqn (36) and an algorithm based on the fundamental theorem of symmetric polynomials [*cf.* Guo *et al.* (1991a)], we can easily calculate $A^{(f)}$, $A^{(0)}$ and $A^{(1)}$, and get $A=(\alpha_{ij})$. The results are as follows:

$$\begin{aligned}\alpha_{11}=\frac{1}{\Delta^2}\Sigma_i(\lambda_j-\lambda_k)\{&f'(\lambda_i)\lambda_j^2\lambda_k^2(\lambda_j-\lambda_k)+\frac{1}{\Delta}f(\lambda_i)[2\,\mathrm{III}(3\,\mathrm{I}\,\mathrm{III}+\mathrm{I}^2\,\mathrm{II}-4\,\mathrm{II}^2)\\ &-(10\,\mathrm{I}\,\mathrm{II}\,\mathrm{III}+\mathrm{I}^2\,\mathrm{II}^2-2\,\mathrm{I}^3\,\mathrm{III}-4\,\mathrm{II}^3-9\,\mathrm{III}^2)(\lambda_j+\lambda_k)]\},\end{aligned}$$

$$\begin{aligned}\alpha_{12}=\frac{1}{\Delta^2}\Sigma_i(\lambda_j-\lambda_k)\{&f'(\lambda_i)\lambda_j\lambda_k(\lambda_k^2-\lambda_j^2)+\frac{1}{\Delta}f(\lambda_i)[2\,\mathrm{III}(7\,\mathrm{I}\,\mathrm{II}-2\,\mathrm{I}^3-9\,\mathrm{III})\\ &-(4\,\mathrm{I}\,\mathrm{II}^2-\mathrm{I}^2\,\mathrm{III}-\mathrm{I}^3\,\mathrm{II}-6\,\mathrm{II}\,\mathrm{III})(\lambda_j+\lambda_k)]\},\end{aligned}$$

$$\begin{aligned}\alpha_{13}=\frac{1}{\Delta^2}\Sigma_i(\lambda_j-\lambda_k)\{&f'(\lambda_i)\lambda_j\lambda_k(\lambda_j-\lambda_k)+\frac{1}{\Delta}f(\lambda_i)[4\,\mathrm{III}(\mathrm{I}^2-3\,\mathrm{II})\\ &-(3\,\mathrm{I}\,\mathrm{III}+\mathrm{I}^2\,\mathrm{II}-4\,\mathrm{II}^2)(\lambda_j+\lambda_k)]\},\end{aligned}$$

$$\begin{aligned}\alpha_{22}=\frac{1}{\Delta^2}\Sigma_i(\lambda_j-\lambda_k)\{&f'(\lambda_i)(\lambda_j+\lambda_k)(\lambda_j^2-\lambda_k^2)\\ &+\frac{1}{\Delta}f(\lambda_i)[2(-3\,\mathrm{I}\,\mathrm{II}^2-\mathrm{I}^2\,\mathrm{III}+\mathrm{I}^3\,\mathrm{II}+3\,\mathrm{II}\,\mathrm{III})\\ &-2(6\,\mathrm{I}\,\mathrm{III}-4\,\mathrm{I}^2\,\mathrm{II}+\mathrm{I}^4+\mathrm{II}^2)(\lambda_j+\lambda_k)]\},\end{aligned}$$

$$\alpha_{23} = \frac{1}{\Delta^2}\Sigma_i(\lambda_j-\lambda_k)\{f'(\lambda_i)(\lambda_k^2-\lambda_j^2)+\frac{1}{\Delta}f(\lambda_i)[2(3\,\mathrm{I}\,\mathrm{III}-\mathrm{I}^2\,\mathrm{II}+2\,\mathrm{II}^2) \\ -(7\,\mathrm{I}\,\mathrm{II}-2\,\mathrm{I}^3-9\,\mathrm{III})(\lambda_j+\lambda_k)]\},$$

$$\alpha_{33} = \frac{1}{\Delta^2}\Sigma_i(\lambda_j-\lambda_k)\{f'\lambda_i)(\lambda_j-\lambda_k)+\frac{1}{\Delta}f(\lambda_i)[2(\mathrm{I}\,\mathrm{II}-9\,\mathrm{III})-2(\mathrm{I}^2-3\,\mathrm{II})(\lambda_j+\lambda_k]\}, \tag{38}$$

where

$$\Delta^2 = 18\,\mathrm{I}\,\mathrm{II}\,\mathrm{III}+\mathrm{I}^2\,\mathrm{II}^2-4\,\mathrm{I}^3\,\mathrm{III}-4\,\mathrm{II}^3-27\,\mathrm{III}^2. \tag{39}$$

Expressing the coefficients α_1, α_2, α_3 in eqn (17) in terms of J_0, J_1, J_2 and (α_{ij}) and taking eqn $(28)_3$ into account, we get a more specific representation of $\dot{\mathbf{E}}$ as follows:

$$\dot{\mathbf{E}}(\mathbf{U};\dot{\mathbf{U}}) = [\alpha_{11}\,\mathrm{tr}\,\dot{\mathbf{U}}+\alpha_{12}\,\mathrm{tr}\,(\mathbf{U}\dot{\mathbf{U}})+\alpha_{13}\,\mathrm{tr}\,(\mathbf{U}^2\dot{\mathbf{U}})]\mathbf{I}+[\alpha_{12}\,\mathrm{tr}\,\dot{\mathbf{U}}+\alpha_{22}\,\mathrm{tr}\,(\mathbf{U}\dot{\mathbf{U}}) \\ +\alpha_{23}\,\mathrm{tr}\,(\mathbf{U}^2\dot{\mathbf{U}})]\mathbf{U}+[\alpha_{13}\,\mathrm{tr}\,\dot{\mathbf{U}}+\alpha_{23}\,\mathrm{tr}\,(\mathbf{U}\dot{\mathbf{U}})+\alpha_{33}\,\mathrm{tr}\,(\mathbf{U}^2\dot{\mathbf{U}})]\mathbf{U}^2 \\ +\alpha_4\,\dot{\mathbf{U}}+\alpha_5(\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}). \tag{40}$$

The final explicit expression for $\dot{\mathbf{E}}$ is obtained by substituting eqns (28) and (38) into eqn (40). Since the distinct eigenvalues λ_1, λ_2 and λ_3 may be expressed explicitly through eqn (18) as functions of the principal invariants I, II and III, the coefficients α_{ij} $(i \leqslant j)$, α_4 and α_5 in eqn (40) are functions of the principal invariants of $\mathbf{U}$.

Let us calculate some examples for illustration. Since the requirements $f(1) = 0$ and $f'(1) = 1$ have no bearing on the validity of our formula, we forgo those requirements in our examples below.

(1) $\mathbf{E} = \mathbf{U}: f(\lambda) = \lambda$,

$$\alpha_4 = 1, \quad \alpha_5 = 0, \quad F = E, \quad A = 0,$$
$$\dot{\mathbf{E}} = \dot{\mathbf{U}}.$$

(2) $\mathbf{E} = \mathbf{U}^2: f(\lambda) = \lambda^2$,

$$\alpha_4 = 0, \quad \alpha_5 = 1, \quad F = 2U, \quad A = 0,$$
$$\dot{\mathbf{E}} = \mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}.$$

These two examples are trivial. The results agree with those obtained from the direct differentiation of $\mathbf{E} = \mathbf{U}$ and $\mathbf{E} = \mathbf{U}\mathbf{U}$, respectively.

(3) $\mathbf{E} = \mathbf{U}^3: f(\lambda) = \lambda^3$,

$$\alpha_4 = -\mathrm{II}, \quad \alpha_5 = \mathrm{I}, \quad F = 3U^2, \quad A = \begin{pmatrix} \mathrm{II} & -\mathrm{I} & 1 \\ -\mathrm{I} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\dot{\mathbf{E}} = [\mathrm{II}\,\mathrm{tr}\,\dot{\mathbf{U}}-\mathrm{I}\,\mathrm{tr}\,(\mathbf{U}\dot{\mathbf{U}})+\mathrm{tr}\,(\mathbf{U}^2\dot{\mathbf{U}})]\mathbf{I}+[-\mathrm{I}\,\mathrm{tr}\,\dot{\mathbf{U}}+\mathrm{tr}\,(\mathbf{U}\dot{\mathbf{U}})]\mathbf{U} \\ +(\mathrm{tr}\,\dot{\mathbf{U}})\mathbf{U}^2-\mathrm{II}\,\dot{\mathbf{U}}+\mathrm{I}(\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}). \tag{41}$$

This formula differs in form from the expression

$$\dot{\mathbf{E}} = \dot{\mathbf{U}}\mathbf{U}^2+\mathbf{U}\dot{\mathbf{U}}\mathbf{U}+\mathbf{U}^2\dot{\mathbf{U}} \tag{42}$$

obtained by the direct differentiation of $\mathbf{E} = \mathbf{U}\mathbf{U}\mathbf{U}$, but it agrees with the result obtained by differentiation of the Cayley–Hamilton equation

$$\mathbf{U}^3 = \mathrm{I}\,\mathbf{U}^2 - \mathrm{II}\,\mathbf{U} + \mathrm{III}\,\mathbf{I},$$

if we additionally use the following formulae on derivatives of principal invariants [*cf.* Carlson and Hoger (1986b) and Guo (1989)]:

$$\dot{\mathrm{I}} = \mathrm{tr}\,\dot{\mathbf{U}}, \quad \dot{\mathrm{II}} = \mathrm{I}\,\mathrm{tr}\,\dot{\mathbf{U}} - \mathrm{tr}\,(\mathbf{U}\dot{\mathbf{U}}),$$

$$\dot{\mathrm{III}} = \mathrm{II}\,\mathrm{tr}\,\dot{\mathbf{U}} - \mathrm{I}\,\mathrm{tr}\,(\mathbf{U}\dot{\mathbf{U}}) + \mathrm{tr}\,(\mathbf{U}^2\dot{\mathbf{U}}).$$

Equation (41) also follows immediately from eqn (42) and Rivlin's identity for the tensor polynomial $\mathbf{A}^2\mathbf{B}+\mathbf{ABA}+\mathbf{AB}^2$ [*cf.* Rivlin (1955)].

(4) $\mathbf{E} = \ln \mathbf{U} : f(\lambda) = \ln \lambda$,

$$\alpha_4 = \frac{1}{\Delta}\Sigma_i \ln \lambda_i(\lambda_j^2 - \lambda_k^2), \quad \alpha_5 = \frac{-1}{\Delta}\Sigma_i \ln \lambda_i(\lambda_j - \lambda_k),$$

$$\alpha_{11} = \frac{1}{\Delta^2}\left[\frac{1}{\mathrm{III}}(14\,\mathrm{I}\,\mathrm{II}^2\,\mathrm{III} + \mathrm{I}^2\,\mathrm{II}^3 - \mathrm{I}^2\,\mathrm{III}^2 - 3\,\mathrm{I}^3\,\mathrm{II}\,\mathrm{III} - 15\,\mathrm{II}\,\mathrm{III}^2 - 4\,\mathrm{II}^4)\right.$$
$$\left. - \alpha_4(10\,\mathrm{I}\,\mathrm{II}\,\mathrm{III} + \mathrm{I}^2\,\mathrm{II}^2 - 2\,\mathrm{I}^3\,\mathrm{III} - 4\,\mathrm{II}^3 - 9\,\mathrm{III}^2) - 2\alpha_5\,\mathrm{III}(3\,\mathrm{I}\,\mathrm{III} + \mathrm{I}^2\,\mathrm{II} - 4\,\mathrm{II}^2)\right],$$

$$\alpha_{12} = \frac{1}{\Delta^2}\left[\frac{1}{\mathrm{III}}(4\,\mathrm{I}\,\mathrm{II}^3 + 12\,\mathrm{I}\,\mathrm{III}^2 - 9\,\mathrm{I}^2\,\mathrm{II}\,\mathrm{III} - \mathrm{I}^3\,\mathrm{II}^2 + 2\,\mathrm{I}^4\,\mathrm{III} - 4\,\mathrm{II}^2\,\mathrm{III})\right.$$
$$\left. - \alpha_4(4\,\mathrm{I}\,\mathrm{II}^2 - \mathrm{I}^2\,\mathrm{III} - \mathrm{I}^3\,\mathrm{II} - 6\,\mathrm{II}\,\mathrm{III}) - 2\alpha_5\,\mathrm{III}(7\,\mathrm{I}\,\mathrm{II} - 2\,\mathrm{I}^3 - 9\,\mathrm{III})\right],$$

$$\alpha_{13} = \frac{1}{\Delta^2}\left[\frac{1}{\mathrm{III}}(10\,\mathrm{I}\,\mathrm{II}\,\mathrm{III} + \mathrm{I}^2\,\mathrm{II}^2 - 2\,\mathrm{I}^3\,\mathrm{III} - 4\,\mathrm{II}^3 - 9\,\mathrm{III}^2)\right.$$
$$\left. - \alpha_4(3\,\mathrm{I}\,\mathrm{III} + \mathrm{I}^2\,\mathrm{II} - 4\,\mathrm{II}^2) - 4\alpha_5\,\mathrm{III}(\mathrm{I}^2 - 3\,\mathrm{II})\right],$$

$$\alpha_{22} = \frac{1}{\Delta^2}\left[\frac{1}{\mathrm{III}}(13\,\mathrm{I}\,\mathrm{II}\,\mathrm{III} - 4\,\mathrm{I}^2\,\mathrm{II}^2 - \mathrm{I}^3\,\mathrm{III} + \mathrm{I}^4\,\mathrm{II} - 9\,\mathrm{III}^2)\right.$$
$$\left. - 2\alpha_4(6\,\mathrm{I}\,\mathrm{III} - 4\,\mathrm{I}^2\,\mathrm{II} + \mathrm{I}^4 + \mathrm{II}^2) - 2\alpha_5(-3\,\mathrm{I}\,\mathrm{II}^2 - \mathrm{I}^2\,\mathrm{III} + \mathrm{I}^3\,\mathrm{II} + 3\,\mathrm{II}\,\mathrm{III})\right],$$

$$\alpha_{23} = \frac{1}{\Delta^2}\left[\frac{1}{\mathrm{III}}(4\,\mathrm{I}\,\mathrm{II}^2 - \mathrm{I}^2\,\mathrm{III} - \mathrm{I}^3\,\mathrm{II} - 6\,\mathrm{II}\,\mathrm{III})\right.$$
$$\left. - \alpha_4(7\,\mathrm{I}\,\mathrm{II} - 2\,\mathrm{I}^3 - 9\,\mathrm{III}) - 2\alpha_5(3\,\mathrm{I}\,\mathrm{III} - \mathrm{I}^2\,\mathrm{II} + 2\,\mathrm{II}^2)\right],$$

$$\alpha_{33} = \frac{1}{\Delta^2}\left[\frac{1}{\mathrm{III}}(3\,\mathrm{I}\,\mathrm{III} + \mathrm{I}^2\,\mathrm{II} - 4\,\mathrm{II}^2) - 2\alpha_4(\mathrm{I}^2 - 3\,\mathrm{II}) - 2\alpha_5(\mathrm{I}\,\mathrm{II} - 9\,\mathrm{III})\right].$$

Substituting these coefficients into eqn (40), we get our formula for $(\ln \mathbf{U})^{\cdot}$. In this formula, except for α_4 and α_5 which involve $\ln \lambda_i$, all the coefficients are already explicitly expressed in terms of the principal invariants I, II, III. With considerable labor, one can obtain Hoger's (1986) formula from the present formula, and vice versa.

4. DOUBLE COALESCENCE

Since $\mathscr{T}_2^0$ is open in $\mathscr{T}$, it is a disjoint union of intervals which are open in $\mathscr{T}$. Our discussion will be local in time, so we may restrict our attention to one of such intervals. As pointed out in Section 2 above, on an interval where the number of distinct eigenvalues remains fixed, there exist C^1 functions $\lambda_i(\cdot)$ which represent the repeated eigenvalues and we may select a C^1 triad of orthonormal eigenvectors $\{\mathbf{N}_i(\cdot)\}$ subordinate to $\{\lambda_i(\cdot)\}$. Without loss of generality, assume

$$\lambda_1(t) \neq \lambda_2(t) = \lambda_3(t) \tag{43}$$

for the interval in question. It follows that the equations

$$\dot{\lambda}_2 = \dot{\lambda}_3, \quad f_2 = f_3, \quad f_{,2} = f_{,3} \tag{44}$$

hold for that interval. Moreover, eqns (5)–(10), (13)–(15) and $(16)_1$ remain valid, whereas eqn $(16)_2$ is valid for $i = 1$ and $j = 2$. In particular, we have

$$\dot{U}_{23} = 0, \tag{45}$$

$$\dot{E}_{22} = \dot{E}_{33}, \quad \dot{E}_{23} = 0, \tag{46}$$

and

$$\left.\begin{aligned} \dot{E}_{11} &= f_{,1}\dot{U}_{11} \\ \dot{E}_{22} &= f_{,2}\dot{U}_{22} \\ \dot{E}_{12} &= \frac{f_1 - f_2}{\lambda_1 - \lambda_2}\dot{U}_{12} \end{aligned}\right\}. \tag{47}$$

This confirms once again the observation of Carlson and Hoger (1986a) that the expression for the rate in the case of double coalescence is of the same form as the one appropriate to the two-dimensional case with distinct eigenvalues.

On the other hand, from the relation $\mathbf{U} = \lambda_2\mathbf{I} + (\lambda_1 - \lambda_2)\mathbf{N}_1 \otimes \mathbf{N}_1$, we obtain

$$\mathbf{U}^2 = \lambda_2^2\mathbf{I} + (\lambda_1^2 - \lambda_2^2)\mathbf{N}_1 \otimes \mathbf{N}_1 = (\lambda_1 + \lambda_2)\mathbf{U} - \lambda_1\lambda_2\mathbf{I} \tag{48}$$

and

$$\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U} = -(\lambda_2\dot{\lambda}_1 + \lambda_1\dot{\lambda}_2)\mathbf{I} + (\dot{\lambda}_1 + \dot{\lambda}_2)\mathbf{U} + (\lambda_1 + \lambda_2)\dot{\mathbf{U}}. \tag{49}$$

Should eqn (17) be valid, substitution of eqns (48) and (49) into eqn (17) would imply that for the case of double coalescence $\dot{\mathbf{E}}$ has the reduced representation

$$\dot{\mathbf{E}}(\mathbf{U}; \dot{\mathbf{U}}) = \beta_1\mathbf{I} + \beta_2\mathbf{U} + \beta_3\dot{\mathbf{U}}, \tag{50}$$

which has only three unknown coefficients. Indeed we shall show by explicit construction that for the present case $\dot{\mathbf{E}}$ can be put in the form (50), where β_3 is a function of

$$\hat{\mathrm{I}} = \lambda_1 + \lambda_2, \quad \hat{\mathrm{II}} = \lambda_1\lambda_2, \tag{51}$$

and β_1, β_2 are functions of $\hat{\mathrm{I}}$, $\hat{\mathrm{II}}$ and of

$$\hat{J}_0 = \dot{\lambda}_1 + \dot{\lambda}_2, \quad \hat{J}_1 = \lambda_1\dot{\lambda}_1 + \lambda_2\dot{\lambda}_2, \tag{52}$$

linear in $\hat{J}_0, \hat{J}_1$.

Analogous to what we did in Section 3, we first put

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = B\begin{pmatrix}\hat{J}_0\\ \hat{J}_1\end{pmatrix} = B\hat{M}^{\mathrm{T}}\begin{pmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\end{pmatrix}, \tag{53}$$

where

$$B := (\beta_{\alpha\beta}), \quad \hat{M} := \begin{pmatrix}1 & \lambda_1\\ 1 & \lambda_2\end{pmatrix}. \tag{54}$$

Here and henceforth, Greek indices range from 1 to 2. A comparison of the componential form of eqn (50) with eqn (47) yields

$$\left.\begin{aligned}\beta_1+\beta_2\lambda_1+\beta_3\dot{U}_{11} &= f_{,1}\dot{U}_{11}\\ \beta_1+\beta_2\lambda_2+\beta_3\dot{U}_{22} &= f_{,2}\dot{U}_{22}\end{aligned}\right\}, \tag{55}$$

$$\beta_3\dot{U}_{12} = \frac{f_1-f_2}{\lambda_1-\lambda_2}\dot{U}_{12}. \tag{56}$$

Because $\dot{U}_{12}$ is arbitrary,

$$\beta_3 = \frac{f_1-f_2}{\lambda_1-\lambda_2}. \tag{57}$$

With the known β_3, eqn (55) reduces to

$$\begin{pmatrix}1 & \lambda_1\\ 1 & \lambda_2\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}f_{,1}-\beta_3 & 0\\ 0 & f_{,2}-\beta_3\end{pmatrix}\begin{pmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\end{pmatrix}. \tag{58}$$

Denoting

$$\hat{F} = \mathrm{diag.}\,(f_{,1}, f_{,2}), \quad \hat{E} = \mathrm{diag.}\,(1,\ 1), \tag{59}$$

and substituting eqns (53), (54) and (59) into eqn (58), we get

$$B = \hat{M}^{-1}(\hat{F}-\beta_3\hat{E})\hat{M}^{-\mathrm{T}}. \tag{60}$$

Making use of

$$\hat{M}^{-1} = \frac{1}{\lambda_2-\lambda_1}\begin{pmatrix}\lambda_2 & -\lambda_1\\ -1 & 1\end{pmatrix},$$

we obtain the following expressions for the entries of B (symmetric):

$$\left.\begin{aligned}\beta_{11} &= \frac{1}{\hat{\Delta}^2}(\Sigma_\alpha f_{,\alpha}\lambda_\beta^2-\beta_3(\hat{\mathrm{I}}^2-2\,\hat{\mathrm{I}}\mathrm{I}))\\ \beta_{12} &= \frac{-1}{\hat{\Delta}^2}(\Sigma_\alpha f_{,\alpha}\lambda_\beta-\beta_3\,\hat{\mathrm{I}})\\ \beta_{22} &= \frac{1}{\hat{\Delta}^2}(\Sigma_\alpha f_{,\alpha}-2\beta_3)\end{aligned}\right\}, \tag{61}$$

where

$$\hat{\Delta}^2 = \hat{\mathrm{I}}^2-4\,\hat{\mathrm{I}}\mathrm{I} = (\lambda_1-\lambda_2)^2. \tag{62}$$

Here and in eqn (63) the summation Σ_α is carried over permutations (α,β) of $(1,2)$. Our final expression for $\dot{\mathbf{E}}$ is

$$\dot{\mathbf{E}}(\mathbf{U};\dot{\mathbf{U}}) = \frac{1}{\hat{\Delta}^2}\{[(\Sigma_\alpha f_{,\alpha}\lambda_\beta^2 - \beta_3(\hat{\mathrm{I}}^2 - 2\,\hat{\mathrm{II}}))\hat{J}_0 - (\Sigma_\alpha f_{,\alpha}\lambda_\beta - \beta_3\hat{\mathrm{I}})\hat{J}_1]\mathbf{I} + [-(\Sigma_\alpha f_{,\alpha}\lambda_\beta - \beta_3\hat{\mathrm{I}})\hat{J}_0 + (\Sigma_\alpha f_{,\alpha} - 2\beta_3)\hat{J}_1]\mathbf{U}\} + \beta_3\dot{\mathbf{U}}. \quad (63)$$

Let us now consider the same examples from the last section.

(1) $\mathbf{E} = \mathbf{U}$: $f(\lambda) = \lambda$, $\beta_3 = 1$, $B = 0$, $\dot{\mathbf{E}} = \dot{\mathbf{U}}$.
(2) $\mathbf{E} = \mathbf{U}^2$: $f(\lambda) = \lambda^2$, $\beta_3 = \hat{\mathrm{I}}$,

$$B = \begin{pmatrix} -\hat{\mathrm{I}} & 1 \\ 1 & 0 \end{pmatrix},$$

$$\dot{\mathbf{E}} = (-\hat{\mathrm{I}}\hat{J}_0 + \hat{J}_1)\mathbf{I} + \hat{J}_0\mathbf{U} + \hat{\mathrm{I}}\dot{\mathbf{U}}.$$

Because of eqn (49), this result is equivalent to

$$\dot{\mathbf{E}} = \mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}$$

as expected.
(3) $\mathbf{E} = \mathbf{U}^3$: $f(\lambda) = \lambda^3$, $\beta_3 = \hat{\mathrm{I}}^2 - \hat{\mathrm{II}}$,

$$B = \begin{pmatrix} -\hat{\mathrm{I}}^2 - \hat{\mathrm{II}} & \hat{\mathrm{I}} \\ \hat{\mathrm{I}} & 1 \end{pmatrix},$$

$$\dot{\mathbf{E}} = [-(\hat{\mathrm{I}}^2 + \hat{\mathrm{II}})\hat{J}_0 + \hat{\mathrm{I}}\hat{J}_1]\mathbf{I} + (\hat{\mathrm{I}}\hat{J}_0 + \hat{J}_1)\mathbf{U} + (\hat{\mathrm{I}}^2 - \hat{\mathrm{II}})\dot{\mathbf{U}}.$$

Taking into account the relations

$$\mathbf{U}\dot{\mathbf{U}}\mathbf{U} = -\hat{\mathrm{II}}\hat{J}_0\mathbf{I} + \hat{J}_1\mathbf{U} + \hat{\mathrm{II}}\dot{\mathbf{U}}, \quad (64)$$

$$\mathbf{U}^2\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^2 = (-\hat{\mathrm{I}}^2\hat{J}_0 + \hat{\mathrm{I}}\hat{J}_1)\mathbf{I} + \hat{\mathrm{I}}\hat{J}_0\mathbf{U} + (\hat{\mathrm{I}}^2 - 2\,\hat{\mathrm{II}})\dot{\mathbf{U}}, \quad (65)$$

which are valid under double coalescence, we can confirm the equivalence of our expression above with eqn (42)

$$\dot{\mathbf{E}} = \mathbf{U}^2\dot{\mathbf{U}} + \mathbf{U}\dot{\mathbf{U}}\mathbf{U} + \mathbf{U}^2\dot{\mathbf{U}}.$$

(4) $\mathbf{E} = \ln\mathbf{U}$: $f(\lambda) = \ln\lambda$, $\beta_3 = \dfrac{\ln(\lambda_1/\lambda_2)}{\lambda_1 - \lambda_2}$,

$$B = \frac{1}{\hat{\Delta}^2\hat{\mathrm{II}}}\begin{pmatrix} \hat{\mathrm{I}}^3 - 3\,\hat{\mathrm{I}}\,\hat{\mathrm{II}} - \beta_3\,\hat{\mathrm{II}}(\hat{\mathrm{I}}^2 - 2\,\hat{\mathrm{II}}) & 2\,\hat{\mathrm{II}} - \hat{\mathrm{I}}^2 + \beta_3\,\hat{\mathrm{I}}\,\hat{\mathrm{II}} \\ 2\,\hat{\mathrm{II}} - \hat{\mathrm{I}}^2 + \beta_3\,\hat{\mathrm{I}}\,\hat{\mathrm{II}} & \hat{\mathrm{I}} - 2\beta_3\hat{\mathrm{II}} \end{pmatrix},$$

$$\dot{\mathbf{E}} = \frac{1}{\hat{\Delta}^2\hat{\mathrm{II}}}\{[(\hat{\mathrm{I}}^3 - 3\,\hat{\mathrm{I}}\,\hat{\mathrm{II}} - \beta_3\,\hat{\mathrm{II}}(\hat{\mathrm{I}}^2 - 2\,\hat{\mathrm{II}}))\hat{J}_0 + (2\,\hat{\mathrm{II}} - \hat{\mathrm{I}}^2 + \beta_3\hat{\mathrm{I}}\,\hat{\mathrm{II}})\hat{J}_1]\mathbf{I} + [(2\,\hat{\mathrm{II}} - \hat{\mathrm{I}}^2 + \beta_3\,\hat{\mathrm{I}}\,\hat{\mathrm{II}})\hat{J}_0 + (\hat{\mathrm{I}} - 2\beta_3\hat{\mathrm{II}})\hat{J}_1]\mathbf{U}\} + \beta_3\dot{\mathbf{U}}. \quad (66)$$

If the formulae (49) and (64) are used to express $\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}$ and $\mathbf{U}\dot{\mathbf{U}}\mathbf{U}$ in terms of $\mathbf{I}$, $\mathbf{U}$ and $\dot{\mathbf{U}}$, then Hoger's (1986) result, written here in the symbols of the present paper, namely

$$(\ln\mathbf{U})^{\cdot} = \frac{1}{\hat{\Delta}^2\hat{\mathrm{II}}}[(\hat{\mathrm{I}} - 2\beta_3\hat{\mathrm{II}})\mathbf{U}\dot{\mathbf{U}}\mathbf{U} + (2\,\hat{\mathrm{II}} - \hat{\mathrm{I}}^2 + \beta_3\,\hat{\mathrm{I}}\,\hat{\mathrm{II}})(\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}) + (\hat{\mathrm{I}}^3 - 3\,\hat{\mathrm{I}}\,\hat{\mathrm{II}} - 2\beta_3\,\hat{\mathrm{II}}^2)\dot{\mathbf{U}}] \quad (67)$$

assumes the form of our expression (66). It can also be verified that by expressing eqn (66) in terms of $\mathbf{U\dot{U}U}$, $\mathbf{U\dot{U}}+\mathbf{\dot{U}U}$ and $\mathbf{\dot{U}}$, one will in turn arrive at Hoger's result (67).

5. TRIPLE COALESCENCE

On $\mathscr{T}_3^0$, the three eigenvalues of $\mathbf{U}$ coalesce, so we have

$$\lambda_1(t) = \lambda_2(t) = \lambda_3(t) := \lambda(t) \tag{68}$$

and

$$f_1 = f_2 = f_3 = f(\lambda). \tag{69}$$

Hence,

$$\mathbf{U}(t) = \lambda(t)\mathbf{I}, \tag{70}$$

$$\mathbf{E}(t) = f(\lambda(t))\mathbf{I}. \tag{71}$$

Since $\mathscr{T}_3^0$ is open in $\mathscr{T}$ and the functions $\lambda(\cdot)$, $f(\cdot)$ are of class C^1, we can differentiate eqns (70) and (71) to obtain

$$\dot{\mathbf{U}}(t) = \dot{\lambda}(t)\mathbf{I}, \tag{72}$$

$$\dot{\mathbf{E}}(t) = f'(\lambda)\dot{\lambda}\mathbf{I}. \tag{73}$$

It follows immediately that

$$\dot{\mathbf{E}} = f'(\lambda)\dot{\mathbf{U}}, \tag{74}$$

which is our basis-free expression for $\dot{\mathbf{E}}$ on $\mathscr{T}_3^0$. Let us now go through, for the present case, the examples considered in Sections 3 and 4 above:

(1) $\mathbf{E} = \mathbf{U}$: $f(\lambda) = \lambda$, $f'(\lambda) = 1$, $\dot{\mathbf{E}} = \dot{\mathbf{U}}$.
(2) $\mathbf{E} = \mathbf{U}^2$: $f(\lambda) = \lambda^2$, $f'(\lambda) = 2\lambda$, $\dot{\mathbf{E}} = 2\lambda\dot{\mathbf{U}}$.
(3) $\mathbf{E} = \mathbf{U}^3$: $f(\lambda) = \lambda^3$, $f'(\lambda) = 3\lambda^2$, $\dot{\mathbf{E}} = 3\lambda^2\dot{\mathbf{U}}$.
(4) $\mathbf{E} = \ln\mathbf{U}$: $f(\lambda) = \ln\lambda$, $f'(\lambda) = 1/\lambda$, $\dot{\mathbf{E}} = (1/\lambda)\dot{\mathbf{U}}$.

By virtue of eqn (70), the end results of examples (2) and (3) are equivalent to $\dot{\mathbf{E}} = \mathbf{U\dot{U}}+\mathbf{\dot{U}U}$ and $\dot{\mathbf{E}} = \mathbf{U}^2\dot{\mathbf{U}}+\mathbf{U\dot{U}U}+\dot{\mathbf{U}}\mathbf{U}^2$, respectively.

6. COMPLETION OF THE FORMULA

In the previous sections we have obtained three basis-free expressions for $\dot{\mathbf{E}}$ valid over the interior $\mathscr{T}_i^0$ of $\mathscr{T}_i$ $(i = 1, 2, 3)$, respectively, where the eigenvalues of $\mathbf{U}$ are distinct, doubly coalescent, and triply coalescent, respectively. To get a complete basis-free formula for $\dot{\mathbf{E}}$, it suffices to derive additional basis-free expressions for $\dot{\mathbf{E}}$ that cover the remaining instants in $\mathscr{T}\backslash(\mathscr{T}_1^0 \cup \mathscr{T}_2^0 \cup \mathscr{T}_3^0)$. In our derivation below, we shall assume that f be of class C^3.

Let us first give a more convenient characterization of the remaining instants. For a subset $\mathscr{A}$ in $\mathscr{T}$, we shall denote the frontier, the complement and the closure of $\mathscr{A}$ in $\mathscr{T}$ by $\mathrm{Fr}(\mathscr{A})$, $\mathscr{A}^c$ and $\overline{\mathscr{A}}$, respectively. As mentioned earlier, we use the relative topology of $\mathscr{T}$ in $\mathbb{R}$. From the definition of the sets $\mathscr{T}_i$ $(i = 1, 2, 3)$ and the continuity of the functions λ_i, it is easy to prove the validity of the following topological assertions:

(1) $\mathscr{T}_1$ is open in $\mathscr{T}$, i.e. $\mathscr{T}_1^0 = \mathscr{T}_1$;
(2) $\mathscr{T}_2^0 = \mathscr{T}_2 \cap (\overline{\mathscr{T}_1})^c$;
(3) $\mathscr{T}_2 = \mathscr{T}_2^0 \cup (\mathscr{T}_2 \cap \overline{\mathscr{T}_1})$;
(4) $\mathscr{T} = \overline{\mathscr{T}_1} \cup \overline{\mathscr{T}_2^0} \cup \mathscr{T}_3^0$.

Since $\overline{\mathscr{T}_1} = \mathscr{T}_1 \cup \mathrm{Fr}(\mathscr{T}_1)$ and $\overline{\mathscr{T}_2^0} = \mathscr{T}_2^0 \cup \mathrm{Fr}(\mathscr{T}_2^0)$, it suffices to find additional basis-free expressions of $\dot{\mathbf{E}}$ for the instants in $\mathrm{Fr}(\mathscr{T}_1)$ and in $\mathrm{Fr}(\mathscr{T}_2^0)$. Because $\mathscr{T}_1$ is open in $\mathscr{T}$, points in $\mathrm{Fr}(\mathscr{T}_1)$ belong either to $\mathscr{T}_2$ or to $\mathscr{T}_3$, and $\mathrm{Fr}(\mathscr{T}_2^0) \cap \mathscr{T}_1$ is empty. By assertion (3) above, if $t \in \mathrm{Fr}(\mathscr{T}_2^0) \cap \mathscr{T}_2$, then $t \in \overline{\mathscr{T}_1}$ because $\mathscr{T}_2^0$ is open in $\mathscr{T}$. Hence, for $t_0 \in \mathscr{T} \backslash (\mathscr{T}_1^0 \cup \mathscr{T}_2^0 \cup \mathscr{T}_3^0)$, we need to consider only the following three possibilities: (i) $t_0 \in \mathrm{Fr}(\mathscr{T}_1) \cap \mathscr{T}_3$; (ii) $t_0 \in \mathrm{Fr}(\mathscr{T}_1) \cap \mathscr{T}_2$; (iii) $t_0 \in \mathrm{Fr}(\mathscr{T}_2^0) \cap \mathscr{T}_3$.

Case (i). $t_0 \in \mathrm{Fr}(\mathscr{T}_1) \cap \mathscr{T}_3$

Let $\lambda_i(t_0) = \lambda$ for $i = 1,2,3$. Let $\{t_n\}$ be a sequence in $\mathscr{T}_1$ which tends to t_0 as $n \to \infty$. Since $\mathbf{U}$, $\dot{\mathbf{U}}$ and $\dot{\mathbf{E}}$ are continuous at $t = t_0$, we shall obtain a basis-free expression of $\dot{\mathbf{E}}(t_0)$ from the equation

$$\dot{\mathbf{E}}(t_n) = \alpha_1(t_n)\mathbf{I} + \alpha_2(t_n)\mathbf{U}(t_n) + \alpha_3(t_n)\mathbf{U}^2(t_n) + \alpha_4(t_n)\dot{\mathbf{U}}(t_n) + \alpha_5(t_n)(\mathbf{U}(t_n)\dot{\mathbf{U}}(t_n) + \dot{\mathbf{U}}(t_n)\mathbf{U}(t_n)) \tag{75}$$

if we can show that $\lim_{n\to\infty} \alpha_p(t_n)$ exists and determine the limit for each $p = 1, \ldots, 5$. Indeed, if we put $l_p = \lim_{n\to\infty} \alpha_p(t_n)$ and use the fact that $\mathbf{U}(t_0) = \lambda\mathbf{I}$, then

$$\dot{\mathbf{E}}(t_0) = (l_1 + l_2\lambda + l_3\lambda^2)\mathbf{I} + (l_4 + 2l_5\lambda)\dot{\mathbf{U}}(t_0). \tag{76}$$

In the analysis below we shall at times restrict our attention to one specific t_n. For brevity, we shall suppress all dependence on t_n whenever no confusion should arise. For instance, it should be clear from the context when $\mathbf{U}$ really means $\mathbf{U}(t_n)$, α_1 means $\alpha_1(t_n)$, etc. A crucial point in our analysis is that we shall derive a different estimate of α_p ($p = 1, \ldots, 5$) for each ordering of the eigenvalues of $\mathbf{U}$. The six estimates for α_p will together guarantee the existence of l_p and deliver its value.

Let us first find l_5. At $t = t_n$, the eigenvalues of $\mathbf{U}$ are distinct; hence they fall into one of the six orderings: $\lambda_3 < \lambda_2 < \lambda_1$, $\lambda_2 < \lambda_3 < \lambda_1$, etc. Without loss of generality, suppose at $t = t_n$ the eigenvalues are in the ordering $\lambda_2 < \lambda_3 < \lambda_1$. We recast eqn $(28)_2$ in the form

$$\alpha_5 = \frac{1}{\lambda_1 - \lambda_2}\left(-\frac{f(\lambda_2) - f(\lambda_3)}{\lambda_2 - \lambda_3} + \frac{f(\lambda_1) - f(\lambda_3)}{\lambda_1 - \lambda_3}\right). \tag{77}$$

By Taylor's theorem and our smoothness assumption on f,

$$\frac{f(\lambda_2) - f(\lambda_3)}{\lambda_2 - \lambda_3} = f'(\lambda_3) + \tfrac{1}{2}f''(\lambda_3 + \theta_2(\lambda_2 - \lambda_3))(\lambda_2 - \lambda_3), \tag{78}$$

$$\frac{f(\lambda_1) - f(\lambda_3)}{\lambda_1 - \lambda_3} = f'(\lambda_3) + \tfrac{1}{2}f''(\lambda_3 + \theta_1(\lambda_1 - \lambda_3))(\lambda_1 - \lambda_3), \tag{79}$$

for some $0 < \theta_j < 1$ ($j = 1,2$). Substituting the preceding equations into eqn (77) and rearranging, we obtain

$$\alpha_5 = \tfrac{1}{2}f''(\lambda_3 + \theta_1(\lambda_1 - \lambda_3)) + \tfrac{1}{2}[f''(\lambda_3 + \theta_1(\lambda_1 - \lambda_3)) - f''(\lambda_3 + \theta_2(\lambda_2 - \lambda_3))]\frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_2}. \tag{80}$$

Since $|\lambda_2 - \lambda_3|/|\lambda_1 - \lambda_2| < 1$ for the given ordering, we deduce the estimate

$$|\alpha_5 - \tfrac{1}{2}f''(\lambda)| \leqslant \tfrac{1}{2}|f''(\lambda) - f''(\lambda_3 + \theta_1(\lambda_1 - \lambda_3))| + \tfrac{1}{2}|f''(\lambda_3 + \theta_1(\lambda_1 - \lambda_3)) - f''(\lambda_3 + \theta_2(\lambda_2 - \lambda_3))|. \tag{81}$$

Should the λ_i be in another ordering, we can recast α_5 into a suitable form similar to eqn

(77) and follow the same procedure to obtain a similar estimate pertaining to that ordering. For instance, for the ordering $\lambda_1 < \lambda_2 < \lambda_3$, we have the estimate

$$|\alpha_5 - \tfrac{1}{2}f''(\lambda)| \leqslant \tfrac{1}{2}|f''(\lambda) - f''(\lambda_2 + \phi_1(\lambda_3 - \lambda_2))| \\ + \tfrac{1}{2}|f''(\lambda_2 + \phi_1(\lambda_3 - \lambda_2)) - f''(\lambda_2 + \phi_2(\lambda_1 - \lambda_2))|, \quad (82)$$

for some $0 < \phi_j < 1$ $(j = 1, 2)$. Thus we have altogether six estimates for α_5, one for each ordering of the eigenvalues.

Observe that if λ_1, λ_2 and λ_3 fall in a δ-neighborhood of λ, then all the arguments of f'' in any of the aforementioned six estimates likewise lie in the same δ-neighborhood of λ. Since f'' is by assumption continuous at λ, we can easily invoke an ε–δ argument to prove that

$$l_5 = \lim_{n\to\infty} \alpha_5(t_n) = \tfrac{1}{2}f''(\lambda). \quad (83)$$

We proceed in a similar way to determine l_4. For the ordering $\lambda_2 < \lambda_3 < \lambda_1$, we recast eqn $(28)_1$ as follows:

$$\alpha_4 = \frac{1}{\lambda_1 - \lambda_2}\left[(\lambda_1 + \lambda_3)\frac{f(\lambda_2) - f(\lambda_3)}{\lambda_2 - \lambda_3} - (\lambda_2 + \lambda_3)\frac{f(\lambda_1) - f(\lambda_3)}{\lambda_1 - \lambda_3}\right]. \quad (84)$$

By using Taylor's theorem and rearranging, we obtain from eqn (84)

$$\begin{aligned} \alpha_4 &= f'(\lambda_3) + \frac{1}{2(\lambda_1 - \lambda_2)}[f''(\lambda_3 + \theta_2(\lambda_2 - \lambda_3))(\lambda_2 - \lambda_3)(\lambda_1 + \lambda_3) \\ &\quad - f''(\lambda_3 + \theta_1(\lambda_1 - \lambda_3))(\lambda_2 + \lambda_3)(\lambda_1 - \lambda_3)] \\ &= f'(\lambda_3) - \lambda_3 f''(\lambda_3 + \theta_2(\lambda_2 - \lambda_3)) \\ &\quad + \tfrac{1}{2}[f''(\lambda_3 + \theta_2(\lambda_2 - \lambda_3)) - f''(\lambda_3 + \theta_1(\lambda_1 - \lambda_3))](\lambda_2 + \lambda_3)\frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}, \end{aligned} \quad (85)$$

for some $0 < \theta_j < 1$ $(j = 1, 2)$. Since $|\lambda_1 - \lambda_3|/|\lambda_1 - \lambda_2| < 1$ for the ordering $\lambda_2 < \lambda_3 < \lambda_1$, we arrive at the estimate

$$|\alpha_4 - (f'(\lambda) - \lambda f''(\lambda))| \leqslant |f'(\lambda_3) - f'(\lambda)| + |\lambda f''(\lambda) - \lambda_3 f''(\lambda_3 + \theta_2(\lambda_2 - \lambda_3))| \\ + \lambda_1 |f''(\lambda_3 + \theta_2(\lambda_2 - \lambda_3)) - f''(\lambda_3 + \theta_1(\lambda_1 - \lambda_3))|. \quad (86)$$

From this estimate and from similar estimates for the other possible orderings, we conclude from the continuity of $f''(\cdot)$ at λ and of $\lambda_i(\cdot)$ at t_0 that

$$l_4 = \lim_{n\to\infty} \alpha_4(t_n) = f'(\lambda) - \lambda f''(\lambda). \quad (87)$$

To determine l_3, first we obtain from the linear system (24) the expression

$$\alpha_3 = \frac{1}{\Delta}\begin{vmatrix} 1 & \lambda_1 & \lambda_1(f'(\lambda_1) - \alpha_4 - 2\alpha_5\lambda_1) \\ 1 & \lambda_2 & \lambda_2(f'(\lambda_2) - \alpha_4 - 2\alpha_5\lambda_2) \\ 1 & \lambda_3 & \lambda_3(f'(\lambda_3) - \alpha_4 - 2\alpha_5\lambda_3) \end{vmatrix} \quad (88)$$

for each t_n. Again we derive an estimate of α_3 for each ordering of the eigenvalues. Since the arguments are similar, we shall present in detail only that which pertains to the ordering $\lambda_2 < \lambda_3 < \lambda_1$. When we expand the numerator determinant in eqn (88) along the third column, α_3 is given as a sum of three terms, each of which is led by a $-\dot{\lambda}_i$ ($i = 1, 2, 3$). For instance, the coefficient of the term led by $-\dot{\lambda}_2$ is

$$\Box_2 := \frac{f'(\lambda_2)-\alpha_4-2\alpha_5\lambda_2}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} = \frac{1}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}\left\{f'(\lambda_2)-\frac{1}{\lambda_1-\lambda_2}\left[(\lambda_1+\lambda_3-2\lambda_2)\frac{f(\lambda_2)-f(\lambda_3)}{\lambda_2-\lambda_3}+(\lambda_2-\lambda_3)\frac{f(\lambda_1)-f(\lambda_3)}{\lambda_1-\lambda_3}\right]\right\}. \tag{89}$$

By Taylor's theorem and the assumption that f is of class C^3, we have

$$\begin{aligned} f'(\lambda_2) &= f'(\lambda_3)+f''(\lambda_3)(\lambda_2-\lambda_3)+\tfrac{1}{2}f'''(\lambda_3+\theta_3(\lambda_2-\lambda_3))(\lambda_2-\lambda_3)^2, \\ \frac{f(\lambda_2)-f(\lambda_3)}{\lambda_2-\lambda_3} &= f'(\lambda_3)+\tfrac{1}{2}f''(\lambda_3)(\lambda_2-\lambda_3)+\tfrac{1}{6}f'''(\lambda_3+\theta_2(\lambda_2-\lambda_3))(\lambda_2-\lambda_3)^2, \\ \frac{f(\lambda_1)-f(\lambda_3)}{\lambda_1-\lambda_3} &= f'(\lambda_3)+\tfrac{1}{2}f''(\lambda_3)(\lambda_1-\lambda_3)+\tfrac{1}{6}f'''(\lambda_3+\theta_1(\lambda_1-\lambda_3))(\lambda_1-\lambda_3)^2, \end{aligned} \tag{90}$$

for some $0 < \theta_j < 1$ ($j = 1, 2, 3$). Substituting these expressions into eqn (89) and rearranging, we obtain

$$\begin{aligned} \Box_2 &= \frac{1}{2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}\Big\{f'''(\lambda_3+\theta_3(\lambda_2-\lambda_3))(\lambda_2-\lambda_3)^2 \\ &\quad -\frac{1}{3(\lambda_1-\lambda_2)}[f'''(\lambda_3+\theta_2(\lambda_2-\lambda_3))(\lambda_1+\lambda_3-2\lambda_2)(\lambda_2-\lambda_3)^2 \\ &\quad +f'''(\lambda_3+\theta_1(\lambda_1-\lambda_3))(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)^2]\Big\} \\ &= -\tfrac{1}{6}f'''(\lambda_3+\theta_1(\lambda_1-\lambda_3))+\tfrac{1}{2}[f'''(\lambda_3+\theta_2(\lambda_2-\lambda_3)) \\ &\quad -f'''(\lambda_3+\theta_1(\lambda_1-\lambda_3))]\frac{\lambda_1+\lambda_3-2\lambda_2}{\lambda_1-\lambda_2}\,\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_2}+\tfrac{1}{2}[f'''(\lambda_3+\theta_3(\lambda_2-\lambda_3)) \\ &\quad -f'''(\lambda_3+\theta_1(\lambda_1-\lambda_3))]\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_2}. \end{aligned} \tag{91}$$

For the ordering $\lambda_2 < \lambda_3 < \lambda_1$, the following inequalities hold:

$$\left|\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_2}\right| < 1, \quad \left|\frac{\lambda_1+\lambda_3-2\lambda_2}{\lambda_1-\lambda_2}\right| < 2. \tag{92}$$

Hence we obtain from eqn (91) the estimate

$$\begin{aligned} |\Box_2-(-\tfrac{1}{6}f'''(\lambda))| &\leqslant \tfrac{1}{6}|f'''(\lambda)-f'''(\lambda_3+\theta_1(\lambda_1-\lambda_3))|+|f'''(\lambda_3+\theta_2(\lambda_2-\lambda_3)) \\ &\quad -f'''(\lambda_3+\theta_1(\lambda_1-\lambda_3))|+\tfrac{1}{2}|f'''(\lambda_3+\theta_3(\lambda_2-\lambda_3))-f'''(\lambda_3+\theta_1(\lambda_1-\lambda_3))|. \end{aligned} \tag{93}$$

By the same procedure, we obtain similar estimates for the coefficients $\Box_1$, $\Box_3$ of the terms led by $-\dot{\lambda}_1$ and $-\dot{\lambda}_3$, respectively. From these estimates and from similar estimates for the other possible orderings of the eigenvalues, we conclude from the continuity of $f'''(\cdot)$ at λ and of $\dot{\lambda}_i(\cdot)$ at t_0 that

$$l_3 = \lim_{n\to\infty}\alpha_3(t_n) = \tfrac{1}{6}(\dot{\lambda}_1+\dot{\lambda}_2+\dot{\lambda}_3)f'''(\lambda). \tag{94}$$

Similarly, from the linear system (24) we obtain the expressions

$$\alpha_1 = \frac{1}{\Delta}\begin{vmatrix} \dot{\lambda}_1(f'(\lambda_1)-\alpha_4-2\alpha_5\lambda_1) & \lambda_1 & \lambda_1^2 \\ \dot{\lambda}_2(f'(\lambda_2)-\alpha_4-2\alpha_5\lambda_2) & \lambda_2 & \lambda_2^2 \\ \dot{\lambda}_3(f'(\lambda_3)-\alpha_4-2\alpha_5\lambda_3) & \lambda_3 & \lambda_3^2 \end{vmatrix} \tag{95}$$

and

$$\alpha_2 = \frac{1}{\Delta}\begin{vmatrix} 1 & \dot{\lambda}_1(f'(\lambda_1)-\alpha_4-2\alpha_5\lambda_1) & \lambda_1^2 \\ 1 & \dot{\lambda}_2(f'(\lambda_2)-\alpha_4-2\alpha_5\lambda_2) & \lambda_2^2 \\ 1 & \dot{\lambda}_3(f'(\lambda_3)-\alpha_4-2\alpha_5\lambda_3) & \lambda_3^2 \end{vmatrix} \tag{96}$$

for each t_n. Starting from these expressions, we conclude from the estimates of $\square_i$ $(i = 1, 2, 3)$ and from the continuity of $f'''(\cdot)$ at λ and of $\lambda_i(\cdot)$, $\dot{\lambda}_i(\cdot)$ at t_0 that

$$l_1 = \lim_{n\to\infty}\alpha_1(t_n) = \tfrac{1}{6}\lambda^2 f'''(\lambda)(\dot{\lambda}_1+\dot{\lambda}_2+\dot{\lambda}_3), \tag{97}$$

$$l_2 = \lim_{n\to\infty}\alpha_2(t_n) = -\tfrac{1}{3}\lambda f'''(\lambda)(\dot{\lambda}_1+\dot{\lambda}_2+\dot{\lambda}_3). \tag{98}$$

Substituting the values of l_j $(j = 1,\ldots,5)$ into eqn (76), we obtain the simple basis-free expression

$$\dot{\mathbf{E}}(t_0) = f'(\lambda)\dot{\mathbf{U}}(t_0). \tag{99}$$

If t_0 is also a cluster point of $\mathcal{T}_3^0$, we can easily infer from eqn (74) the validity of expression (99) for that special case. But, even if t_0 does not belong to $\overline{\mathcal{T}_3^0}$, the proof above has shown that the same expression remains valid so long as $t_0 \in \mathrm{Fr}(\mathcal{T}_1)\cap\mathcal{T}_3$.

Case (ii). $t_0 \in \mathrm{Fr}(\mathcal{T}_1)\cap\mathcal{T}_2$

Suppose, without loss of generality, that $\lambda_2(t_0) = \lambda_3(t_0) = \xi$. Let $\{t_n\}$ be a sequence in $\mathcal{T}_1$ that tends to t_0 as $n\to\infty$. If $l_p^{(1)} = \lim_{n\to\infty}\alpha_p(t_n)$ exists for each $p = 1,\ldots,5$, then we shall obtain from eqn (75) a basis-free expression for $\dot{\mathbf{E}}(t_0)$. By eqn (77) and the mean-value theorem, at $t = t_n$,

$$\alpha_5 = \frac{1}{\lambda_1-\lambda_2}\left[-f'(\lambda_3+\theta(t_n)(\lambda_2-\lambda_3))+\frac{f(\lambda_1)-f(\lambda_3)}{\lambda_1-\lambda_3}\right] \tag{100}$$

for some $0 < \theta(t_n) < 1$. Hence

$$l_5^{(1)} = \lim_{n\to\infty}\alpha_5(t_n) = \frac{1}{\lambda_1-\xi}\left(-f'(\xi)+\frac{f(\lambda_1)-f(\xi)}{\lambda_1-\xi}\right), \tag{101}$$

where we have written λ_1 for $\lambda_1(t_0)$. Similarly, starting from eqns (84), (88), (95) and (96), we easily obtain the following formulae:

$$l_4^{(1)} = \frac{1}{\lambda_1-\xi}\left[(\lambda_1+\xi)f'(\xi)-2\xi\frac{f(\lambda_1)-f(\xi)}{\lambda_1-\xi}\right], \tag{102}$$

$$l_3^{(1)} = \frac{1}{(\lambda_1-\xi)^2}\left[\dot{\lambda}_1 f'(\lambda_1)+(\dot{\lambda}_1-\dot{\lambda}_2-\dot{\lambda}_3)f'(\xi) -(2\dot{\lambda}_1-\dot{\lambda}_2-\dot{\lambda}_3)\frac{f(\lambda_1)-f(\xi)}{\lambda_1-\xi}-\tfrac{1}{2}(\dot{\lambda}_2+\dot{\lambda}_3)(\lambda_1-\xi)f''(\xi)\right], \tag{103}$$

$$l_2^{(1)} = -\frac{2}{(\lambda_1-\xi)^2}\left\{\dot{\lambda}_1\xi f'(\lambda_1)+[\dot{\lambda}_1\xi-\tfrac{1}{2}(\lambda_1+\xi)(\dot{\lambda}_2+\dot{\lambda}_3)]f'(\xi)\right.$$
$$\left.-[2\dot{\lambda}_1\xi-\tfrac{1}{2}(\lambda_1+\xi)(\dot{\lambda}_2+\dot{\lambda}_3)]\frac{f(\lambda_1)-f(\xi)}{\lambda_1-\xi}-\tfrac{1}{4}(\lambda_1^2-\xi^2)(\dot{\lambda}_2+\dot{\lambda}_3)f''(\xi)\right\}, \quad (104)$$

$$l_1^{(1)} = \frac{1}{(\lambda_1-\xi)^2}\left\{\dot{\lambda}_1\xi^2 f'(\lambda_1)+\xi[\dot{\lambda}_1\xi-\lambda_1(\dot{\lambda}_2+\dot{\lambda}_3)]f'(\xi)\right.$$
$$\left.-\xi[2\dot{\lambda}_1\xi-\lambda_1(\dot{\lambda}_2+\dot{\lambda}_3)]\frac{f(\lambda_1)-f(\xi)}{\lambda_1-\xi}-\tfrac{1}{2}\lambda_1\xi(\lambda_1-\xi)(\dot{\lambda}_2+\dot{\lambda}_3)f''(\xi)\right\}, \quad (105)$$

where λ_1 and $\dot{\lambda}_i$ $(i = 1,2,3)$ stand for $\lambda_1(t_0)$ and $\dot{\lambda}_i(t_0)$, respectively. Similarly, we obtain the corresponding formulae for the limits $l_p^{(2)}$ and $l_p^{(3)}$ $(p = 1,\ldots,5)$ when $\lambda_3(t_0) = \lambda_1(t_0) = \xi \neq \lambda_2(t_0)$ and $\lambda_1(t_0) = \lambda_2(t_0) = \xi \neq \lambda_3(t_0)$, respectively.

Substitution of the expressions for $l_p^{(k)}$ into the equation

$$\dot{\mathbf{E}}(t_0) = l_1^{(k)}\mathbf{I}+l_2^{(k)}\mathbf{U}+l_3^{(k)}\mathbf{U}^2+l_4^{(k)}\dot{\mathbf{U}}+l_5^{(k)}(\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}) \quad (106)$$

results in a basis-free expression for $\dot{\mathbf{E}}(t_0)$ appropriate to the subcase in question.

Suppose now t_0 is also a cluster point of $\mathscr{T}_2^0$. The set $\mathscr{T}_2^0$ is clearly the disjoint union of three open subsets $\mathscr{A}_i$ defined by $\lambda_i \neq \lambda_j = \lambda_k$, where $i \neq j \neq k \neq i$ $(i,j,k = 1,2,3)$. Without loss of generality, let $\lambda_2(t_0) = \lambda_3(t_0) = \xi \neq \lambda_1(t_0)$. Then t_0 must only be a cluster point of $\mathscr{A}_1$, and $\dot{\lambda}_2(t_0) = \dot{\lambda}_3(t_0)$. Let us write $\dot{\xi} = \dot{\lambda}_2(t_0) = \dot{\lambda}_3(t_0)$. For $t \in \mathscr{A}_1$, by eqns (48) and (49), we have

$$\mathbf{U}^2 = (\lambda_1+\lambda_2)\mathbf{U}-\lambda_1\lambda_2\mathbf{I} \quad (107)$$

and

$$\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U} = (\dot{\lambda}_1+\dot{\lambda}_2)\mathbf{U}+(\lambda_1+\lambda_2)\dot{\mathbf{U}}-(\lambda_1\dot{\lambda}_2+\lambda_2\dot{\lambda}_1)\mathbf{I}. \quad (108)$$

By the continuity of λ_1, λ_2, $\dot{\lambda}_1$, $\dot{\lambda}_2$, $\mathbf{U}$ and $\dot{\mathbf{U}}$, and by the presumption that t_0 is a cluster point of $\mathscr{A}_1$, we conclude that at $t = t_0$,

$$\mathbf{U}^2 = (\lambda_1+\xi)\mathbf{U}-\lambda_1\xi\mathbf{I}, \quad (109)$$

$$\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U} = (\dot{\lambda}_1+\dot{\xi})\mathbf{U}+(\lambda_1+\xi)\dot{\mathbf{U}}-(\lambda_1\dot{\xi}+\xi\dot{\lambda}_1)\mathbf{I}. \quad (110)$$

Substituting the preceding equations into eqn (106) for $k = 1$ and putting $\dot{\lambda}_2 = \dot{\lambda}_3 = \dot{\xi}$ in $l_p^{(1)}$ $(p = 1,\ldots,5)$, we arrive at the expression

$$\dot{\mathbf{E}}(t_0) = \frac{1}{\lambda_1-\xi}\left\{\left[-\xi\dot{\lambda}_1 f'(\lambda_1)+\lambda_1\dot{\xi}f'(\xi)+(\xi\dot{\lambda}_1-\lambda_1\dot{\xi})\frac{f(\lambda_1)-f(\xi)}{\lambda_1-\xi}\right]\mathbf{I}\right.$$
$$\left.+\left[\dot{\lambda}_1 f'(\lambda_1)-\dot{\xi}f'(\xi)-(\dot{\lambda}_1-\dot{\xi})\frac{f(\lambda_1)-f(\xi)}{\lambda_1-\xi}\right]\mathbf{U}+[f(\lambda_1)-f(\xi)]\dot{\mathbf{U}}\right\}, \quad (111)$$

which agrees with what we obtain from eqn (63) under the assumption that t_0 is a cluster point of $\mathscr{A}_1$.

Case (iii). $t_0 \in \mathrm{Fr}(\mathscr{T}_2^0) \cap \mathscr{T}_3$

Without loss of generality, suppose t_0 is a cluster point of $\mathscr{A}_1$. Let $\lambda_i(t_0) = \lambda$ for $i = 1, 2, 3$. For the coefficients β_1, β_2 and β_3 in eqn (51), we can use the corresponding expressions in eqn (111) with the understanding that here $\xi = \lambda_2(t) = \lambda_3(t)$ and $\dot{\xi} = \dot{\lambda}_2(t) = \dot{\lambda}_3(t)$. Let $\{t_n\}$ be a sequence in $\mathscr{A}_1$ that tends to t_0 as $n \to \infty$. By Taylor's theorem and our smoothness assumption on f, we find for each t_n

$$\beta_1 = \frac{\xi\dot{\lambda}_1}{2}[f''(\xi+\theta_2(t_n)(\lambda_1-\xi))-f''(\xi+\theta_1(t_n)(\lambda_1-\xi))] \\ -\frac{\lambda_1\dot{\xi}}{2}f''(\xi+\theta_2(t_n)(\lambda_1-\xi))-\frac{\xi\dot{\lambda}_1}{2}f''(\xi+\theta_1(t_n)(\lambda_1-\xi)), \quad (112)$$

$$\beta_2 = \frac{\dot{\lambda}_1}{2}[f''(\xi+\phi_2(t_n)(\lambda_1-\xi))-f''(\xi+\phi_1(t_n)(\lambda_1-\xi))] \\ +\frac{\dot{\xi}}{2}f''(\xi+\phi_1(t_n)(\lambda_1-\xi))+\frac{\dot{\lambda}_1}{2}f''(\xi+\phi_2(t_n)(\lambda_1-\xi)), \quad (113)$$

$$\beta_3 = f'(\xi+\zeta(t_n)(\lambda_1-\xi)), \quad (114)$$

for some $0 < \theta_j(t_n) < 1$, $0 < \phi_j(t_n) < 1$ and $0 < \zeta(t_n) < 1$ $(j = 1, 2)$. It then follows from the continuity of λ_1, ξ, $\dot{\lambda}_1$, $\dot{\xi}$ at $t = t_0$ and of f', f'' at λ that

$$\lim_{n\to\infty} \beta_1(t_n) = -\tfrac{1}{2}\lambda(\dot{\lambda}_1+\dot{\xi})f''(\lambda), \quad (115)$$

$$\lim_{n\to\infty} \beta_2(t_n) = \tfrac{1}{2}(\dot{\lambda}_1+\dot{\xi})f''(\lambda), \quad (116)$$

$$\lim_{n\to\infty} \beta_3(t_n) = f'(\lambda). \quad (117)$$

Since $\mathbf{U} = \lambda\mathbf{I}$ at $t = t_0$, we conclude from eqn (50) that

$$\dot{\mathbf{E}}(t_0) = f'(\lambda)\dot{\mathbf{U}}. \quad (118)$$

Gathering eqns (74), (99) and (118), we observe that the simple formula $\dot{\mathbf{E}}(t) = f'(\lambda)\dot{\mathbf{U}}$ is valid whenever all the eigenvalues coalesce at the instant t, irrespective of whether $\dot{\mathbf{U}}(t)$ is a spherical tensor. All these amount to saying that

$$\mathbf{DE}(\lambda\mathbf{I})[\dot{\mathbf{U}}] = f'(\lambda)\dot{\mathbf{U}}, \quad (119)$$

where $\mathbf{DE}(\lambda\mathbf{I})$ denotes the derivative of $\mathbf{E}$ at $\mathbf{U} = \lambda\mathbf{I}$. Formula (119) is well known, although a rigorous proof of it is harder to come by.

7. CONTINUOUS EXTENSION OF THE FUNCTIONS α_i

The basis-free formula derived in Sections 3–6 above may be presented in another way.

The functions α_p, as given in eqns (28), (88), (95) and (96), are defined only when the eigenvalues of $\mathbf{U}$ are distinct. Now, let us put

$$\alpha_p = \begin{cases} l_p & \text{if } \lambda_1 = \lambda_2 = \lambda_3 = \lambda, \\ l_p^{(k)} & \text{if } \lambda_i = \lambda_j \neq \lambda_k, \end{cases} \quad (120)$$

$(p = 1, \ldots, 5;\ i, j, k = 1, 2, 3;\ i \neq j \neq k \neq i)$; the functions l_p are given by eqns (83), (87), (94), (97) and (98); the functions $l_p^{(k)}$ are given by eqns (101)–(105) for $k = 1$, and by a cyclic change of indices in those equations for $k = 2, 3$, respectively. We claim that the functions α_p, with their domains thus extended, are continuous in the variables λ_1, λ_2 and λ_3 over the set $\{(\lambda_1, \lambda_2, \lambda_3) : \lambda_i > 0,\ i = 1, 2, 3\}$.

To substantiate this claim, it suffices to prove the continuity of α_p $(p = 1, \ldots, 5)$ at points of the form $(\lambda, \lambda, \lambda)$ and $(\lambda_1, \lambda, \lambda)$, where $\lambda_1 \neq \lambda$. Proofs of continuity of α_p at points of the form $(\lambda, \lambda_2, \lambda)$ and $(\lambda, \lambda, \lambda_3)$, where $\lambda_2 \neq \lambda$ and $\lambda_3 \neq \lambda$, are similar.

Continuity at points of the form $(\lambda, \lambda, \lambda)$

We first establish the continuity of the extended function α_5. If $\lambda_1 = \lambda_2 = \lambda_3 = \xi$, we have

$$|\alpha_5(\xi, \xi, \xi) - \alpha_5(\lambda, \lambda, \lambda)| = \tfrac{1}{2}|f''(\xi) - f''(\lambda)|. \tag{121}$$

If $\lambda_2 = \lambda_3 = \xi$, then by definition

$$\begin{aligned} \alpha_5(\lambda_1, \xi, \xi) = l_5^{(1)} &= \frac{1}{\lambda_1 - \xi}\left(-f'(\xi) + \frac{f(\lambda_1) - f(\xi)}{\lambda_1 - \xi}\right) \\ &= \tfrac{1}{2} f''(\xi + \theta(\lambda_1 - \xi)) \end{aligned} \tag{122}$$

for some $0 < \theta < 1$. Hence

$$|\alpha_5(\lambda_1, \xi, \xi) - \alpha_5(\lambda, \lambda, \lambda)| = \tfrac{1}{2}|f''(\xi + \theta(\lambda_1 - \xi)) - f''(\lambda)|, \tag{123}$$

and $\xi + \theta(\lambda_1 - \xi)$ will fall in a δ-neighborhood of λ if both $|\lambda_1 - \lambda| < \delta$ and $|\xi - \lambda| < \delta$. Analogues of eqn (123) are easily obtained for $|\alpha_5(\xi, \lambda_2, \xi) - \alpha_5(\lambda, \lambda, \lambda)|$ and $|\alpha_5(\xi, \xi, \lambda_3) - \alpha_5(\lambda, \lambda, \lambda)|$, respectively.

Equation (121), eqn (123) and its analogues, together with the six estimates [cf. eqns (81) and (82)] for α_5 obtained earlier show that the extended function α_5 is continuous at the point $(\lambda, \lambda, \lambda)$.

Our proof of the continuity of α_4 at $(\lambda, \lambda, \lambda)$ is similar.

Let us proceed to prove the continuity of α_3 at $(\lambda, \lambda, \lambda)$. For brevity, we write $\alpha_3(\lambda_1, \lambda_2, \lambda_3;\ \dot{\lambda}_i)$ for $\alpha_3(\lambda_1, \lambda_2, \lambda_3, \dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3)$. If $\lambda_1 = \lambda_2 = \lambda_3 = \xi$, we have

$$|\alpha_3(\xi, \xi, \xi;\ \dot{\lambda}_i) - \alpha_3(\lambda, \lambda, \lambda;\ \dot{\lambda}_i)| = \tfrac{1}{6}|\dot{\lambda}_1 + \dot{\lambda}_2 + \dot{\lambda}_3||f'''(\lambda) - f'''(\xi)|. \tag{124}$$

If $\lambda_2 = \lambda_3 = \xi$, then by Taylor's theorem we obtain from eqn (103)

$$\alpha_3(\lambda_1, \xi, \xi;\ \dot{\lambda}_i) = \tfrac{1}{6}[\dot{\lambda}_1 f'''(\xi + \theta_1(\lambda_1 - \xi)) + (\dot{\lambda}_2 + \dot{\lambda}_3) f'''(\xi + \theta_2(\lambda_1 - \xi))], \tag{125}$$

for some $0 < \theta_j < 1$ $(j = 1, 2)$. Hence we obtain the estimate

$$|\alpha_3(\lambda_1, \xi, \xi;\ \dot{\lambda}_i) - \alpha_3(\lambda, \lambda, \lambda;\ \dot{\lambda}_i)| \leqslant \tfrac{1}{6}\textstyle\sum_{j=1}^{3} |\dot{\lambda}_j||f'''(\xi + \theta_j(\lambda_1 - \xi)) - f'''(\lambda)|, \tag{126}$$

where $\theta_3 = \theta_2$. Similar estimates can easily be obtained for $|\alpha_3(\xi, \lambda_2, \xi;\ \dot{\lambda}_i) - \alpha_3(\lambda, \lambda, \lambda;\ \dot{\lambda}_i)|$ and $|\alpha_3(\xi, \xi, \lambda_3;\ \dot{\lambda}_i) - \alpha_3(\lambda, \lambda, \lambda;\ \dot{\lambda}_i)|$, respectively. Gathering these estimates with eqn (124) and the six estimates obtained earlier [*cf.* eqn (93)], we see that $\alpha_3(\cdot, \cdot, \cdot;\ \dot{\lambda}_i)$ is continuous at the point $(\lambda, \lambda, \lambda)$. Proofs of continuity of $\alpha_1(\cdot, \cdot, \cdot;\ \dot{\lambda}_i)$ and $\alpha_2(\cdot, \cdot, \cdot;\ \dot{\lambda}_i)$ at $(\lambda, \lambda, \lambda)$ are similar.

Continuity at points of the form $(\lambda_1, \lambda, \lambda)$, *where* $\lambda_1 \neq \lambda$

Now we show that the extended functions α_p are continuous at points of the form $(\lambda_1, \lambda, \lambda)$, where $\lambda_1 \neq \lambda$. To this end, instead of writing down explicit estimates, it is easier to follow a line of argument used by Serrin (1959) and by Carlson and Hoger (1986a) in similar problems.

Let $(\bar{\lambda}_1, \xi/\sqrt{2}, \xi/\sqrt{2})$ be the image of the orthogonal projection of the point $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$ onto the $\lambda_2 = \lambda_3$ plane. Then

$$\bar{\lambda}_2 = \frac{1}{\sqrt{2}}(\xi - d), \tag{127}$$

$$\bar{\lambda}_3 = \frac{1}{\sqrt{2}}(\xi + d), \tag{128}$$

where $|d|$ is the distance between $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$ and $(\bar{\lambda}_1, \xi/\sqrt{2}, \xi/\sqrt{2})$. We use $((\bar{\lambda}_1, \xi), d)$ as variables and treat the λ_1–λ_2–λ_3 space as the Cartesian product $\Pi_1 \times \Pi_1^\perp$, where Π_1 denotes the $\lambda_2 = \lambda_3$ plane and $\Pi_1^\perp$ its orthogonal complement.

With this change of variables, we shall consider the continuity of α_5 at $((\lambda_1, \sqrt{2}\lambda), 0)$, where $\lambda_1 \neq \lambda$. It is easy to show that

$$\lim_{d \to 0} \alpha_5((\bar{\lambda}_1, \xi), d) = \alpha_5((\bar{\lambda}_1, \xi), 0) \tag{129}$$

is subuniform with respect to $(\bar{\lambda}_1, \xi)$, and

$$\lim_{(\bar{\lambda}_1, \xi) \to (\lambda_1, \sqrt{2}\lambda)} \alpha_5((\bar{\lambda}_1, \xi), d) = \alpha_5((\lambda_1, \sqrt{2}\lambda), d) \tag{130}$$

exists pointwise for each sufficiently small d. Hence by the Moore–Osgood theorem [*cf.* Munroe (1965)], the "double-limit"

$$\lim_{((\bar{\lambda}_1, \xi), d) \to ((\lambda_1, \sqrt{2}\lambda), 0)} \alpha_5 \tag{131}$$

exists and is equal to the iterated limit

$$\lim_{(\bar{\lambda}_1, \xi) \to (\lambda_1, \sqrt{2}\lambda)} \left(\lim_{d \to 0} \alpha_5\right) = \alpha_5((\lambda_1, \sqrt{2}\lambda), 0) = \alpha_5(\lambda_1, \lambda, \lambda), \tag{132}$$

where we have returned to using the variables $(\lambda_1, \lambda_2, \lambda_3)$ when we write $(\lambda_1, \lambda, \lambda)$ for $((\lambda_1, \sqrt{2}\lambda), 0)$. Hence

$$\lim_{(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) \to (\lambda_1, \lambda, \lambda)} \alpha_5 = \alpha_5(\lambda_1, \lambda, \lambda), \tag{133}$$

or the extended function α_5 is continuous at the point $(\lambda_1, \lambda, \lambda)$. Similarly we can establish the continuity of α_p $(p = 1, \ldots, 4)$ at points of the form $(\lambda_1, \lambda, \lambda)$.

Now consider the equation

$$\dot{\mathbf{E}} = \alpha_1 \mathbf{I} + \alpha_2 \mathbf{U} + \alpha_3 \mathbf{U}^2 + \alpha_4 \dot{\mathbf{U}} + \alpha_5(\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}), \tag{134}$$

which gives a representation of $\dot{\mathbf{E}}$ when the eigenvalues of $\mathbf{U}$ are distinct (*cf.* Section 3 above). After we extend the domains of α_p by means of the definition (120), the right-hand side of eqn (134) becomes a continuous function of the time t when the continuous functions $\lambda_i(\cdot)$, $\dot{\lambda}_i(\cdot)$, $\mathbf{U}(\cdot)$ and $\dot{\mathbf{U}}(\cdot)$ are substituted in, with no restriction whatsoever on the coalescence of principal stretches. From definition (120) and our discussions in Sections 3–6 above, it is clear that eqn (134), with the coefficients α_p extended by continuity, will give a basis-free representation of $\dot{\mathbf{E}}$ for each t in the interval $\mathscr{T}$. Indeed this representation is nothing but the same formula derived in Sections 3–6 above. Here it is presented in a deceivingly compact form; all the complicated expressions are now swept behind the definition of the extended α_p $(p = 1, \ldots, 5)$.

Remark 7.1. While α_i ($i = 1, 2, 3$) can be extended by continuity to allow for coalescence of principal stretches, it does not immediately follow that α_{ij} ($i \leqslant j$), as given in eqn (38), can likewise be extended. We surmise such extensions could be done, provided that f is sufficiently smooth, but we elect not to pursue this issue further in the present paper. □

Remark 7.2. In proving the validity of their formula for the derivative **DE** of **E**, Carlson and Hoger (1986a) faced the same problem as ours in showing the continuity of scalar coefficients such as our α_p at points of the form $(\lambda, \lambda, \lambda)$, $(\lambda_1, \lambda, \lambda)$, etc. Their proof had a slip in the part that dealt with points of the form $(\lambda, \lambda, \lambda)$. Let us describe briefly what they did.

In their paper Carlson and Hoger give a complete proof of their formula for **DE** in the two-dimensional case. For the three-dimensional case, which is the concern of our present paper, they do not show their proof that the scalar coefficients in their formula for **DE** can be extended by continuity, but assert that "the limits can be calculated exactly" as in their discussion on the continuous extension of the scalar coefficients in their formula for **E** (see their Section 3.3; their **F** is our **E**). Their formula for **E** has three scalar coefficients a, b and c, where a and b are none other than our α_5 and α_4, respectively, and

$$c = \frac{1}{\Delta}\begin{vmatrix} f(\lambda_1) & \lambda_1 & \lambda_1^2 \\ f(\lambda_2) & \lambda_2 & \lambda_2^2 \\ f(\lambda_3) & \lambda_3 & \lambda_3^2 \end{vmatrix}. \tag{135}$$

Let (ξ, ξ, ξ) be the image of the orthogonal projection of $(\lambda_1, \lambda_2, \lambda_3)$ on the line $\lambda_1 = \lambda_2 = \lambda_3$. Carlson and Hoger determine, for each of their scalar coefficients a, b and c, the iterated limit as $(\lambda_1, \lambda_2, \lambda_3) \to (\xi, \xi, \xi)$ along the line normal to the line $\lambda_1 = \lambda_2 = \lambda_3$, followed by $(\xi, \xi, \xi) \to (\lambda, \lambda, \lambda)$ along the line $\lambda_1 = \lambda_2 = \lambda_3$. For each of the scalar coefficients, they mention that the first limit is uniform with respect to ξ and derive a formula [*cf.* their eqn (3.37)] in terms of ξ, $f(\xi)$, $f'(\xi)$ and $f''(\xi)$ with coefficients expressed in d_1, d_2 and d_3, which are the components of the unit vector directed from (ξ, ξ, ξ) toward $(\lambda_1, \lambda_2, \lambda_3)$. Since all the coefficients in Carlson and Hoger's formulae are ratios of two alternating polynomials of degree 3 in d_i ($i = 1, 2, 3$), they are in fact independent of d_i. Thus Carlson and Hoger, whether they noticed it then or not, in effect showed that the limiting values of the scalar coefficients a, b and c, as $(\lambda_1, \lambda_2, \lambda_3) \to (\xi, \xi, \xi)$ along the line normal to the line $\lambda_1 = \lambda_2 = \lambda_3$, are independent of the direction unit vector which points from (ξ, ξ, ξ) toward $(\lambda_1, \lambda_2, \lambda_3)$. But all these as yet do not suffice to guarantee the existence of the limits of their scalar coefficients as $(\lambda_1, \lambda_2, \lambda_3) \to (\lambda, \lambda, \lambda)$ in the three-dimensional λ_1–λ_2–λ_3 space. Indeed counterexamples to this effect can easily be constructed by modifying examples in textbooks of calculus that illustrate the difference between double limits and limits along lines in $\mathbb{R}^2$ (e.g. the continuous function $g: \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^1$ defined by $g(x, y) = x^2y/(x^4+y^2)$ has no limit at the origin, but $g(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any straight line through the origin).

Their proof for continuous extension of the coefficients a, b and c can be completed by the method we use above. Indeed a closer examination of our work on α_4 and α_5 reveals that if f is of class C^2, then $a = \alpha_5$ and $b = \alpha_4$ can be extended by continuity to allow for coalescence of principal stretches. By using a similar argument, we can show that their coefficient c can likewise be extended by continuity if f is of class C^2. It seems likely that our method of proof can be adapted also to treat the scalar coefficients in their basis-free formula for **DE**.

In their theorem on the continuity of the scalar coefficients in their formula for **E**, they impose the sufficient condition that f be of class C^3. Our proof shows that their theorem will be valid if f is of class C^2. This observation suggests that in the theorem where they present their basis-free formula [their eqn (3.4.1)] for **DE**, their smoothness condition on f may be unnecessarily strong. In other words, the condition that f be of class C^7 could likely be weakened as well.

8. CONCLUDING REMARKS

Once one basis-free formula has been obtained by Carlson and Hoger (1986a), other basis-free formulae for $\dot{\mathbf{E}}$ are certainly possible. The basis-free formula presented in this paper, however, is derived by a method totally different than that of Carlson and Hoger's. The two basis-free formulae of course deliver the same $\dot{\mathbf{E}}$ when they are both applicable. To ensure the validity of our formula, however, it suffices that f be of class C^3. On the other hand, Carlson and Hoger asserted for their formula the sufficient condition that f be of class C^7. One difference between the two formulae may lie in the different requirement that their validity puts on the smoothness of f.

As pointed out in Remark 7.2 above, there is a gap in Carlson and Hoger's proof of their formula. We believe the method that we employ to prove the continuity of the extended coefficient functions α_p can be used to complete their proof, and we expect a smoothness assumption on f weaker than C^7 would suffice for the proof.

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